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STRUCTURAL SYNTHESIS OF SYMMETRIC WAFFLE PLATE

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SUMMARY

Structural synthesis has been defined as the rational directed evolution of a structural configuration, which in terms of a defined criterion, efficiently performs a set of specified functional purposes. Structural synthesis is essentially a problem in the programming of interdependent activities involving three types of considerations; namely, a specified set of requirements, a given technology and a criterion by means of which choices can be made between various designs.

The structural configuration employed as an example in this study is an integrally stiffened waffle plate used primarily for aero-space structures. The criterion of design selection employed is the total weight of the waffle. (See Figure 1)

The synthesis technique developed as a result of this research provides a starting point for the optimization of other engineering systems which have the following characteristics:

a. relative minima
b. non-linear inequality constraints
c. a multitude of side constraints
d. non-linear merit function

This note reports the successful development of a synthesis capability for symmetric waffle plates based on the technology presented herein.

INTRODUCTION

Analysis as a tool of structural design is well known. However, the really effective use of an analysis requires that rational methods of directed redesign be developed. As with any structural system, there has been a tendency to regard the problem as being solved when a reliable method of analysis has been developed, while in fact the availability of a reliable method of analysis is only a prerequisite to tackling the task of design synthesis.

Structural synthesis has been defined as the rational directed evolution of a structural configuration which, in terms of a defined criterion, efficiently performs a set of specified functional purposes. Structural synthesis is essentially a problem in the programming of interdependent activities involving three types of considerations; namely, a specified set of requirements, a given technology and a criterion by means of which choices can be made between various designs.

The structural design cycle can be thought of in terms of three main phases:

a. Establish a trial design consistent with the requirements.
b. Carry out an analysis based on this trial design using the accepted technology.
c. Based on the analysis, modify the trial design such that the merit function is improved.

In the past, the redesign phase has been based primarily on an artful combination of experience, judgment and often courage. Consequently the redesign process is not clearly defined. Also, the number of trips around the design cycle has been limited by the available manpower and the time required. The huge strides that have been made in the digital computing field have substantially reduced the time to complete one design cycle. The problem of stating mathematically the philosophy or basis on which redesign decisions are made is the major obstacle to the development of methods of structural synthesis.

Specifications and Requirements

The design load system is made up of several sets of mechanical and thermal loads. When the design load system involves a multiplicity of load conditions, the minimum weight optimum design will be a balanced design for the entire design load system. It should be recognized that if an optimum design is sought using each design load condition separately, several distinct incompatible designs will result.

The basic requirement of the structural system is that it must maintain its structural integrity while being subject to the design load system. The design is inadequate and the structure is said to fail if the structural behavior does not remain within the confines of the stated limits. What constitutes failure must be carefully defined and this can be expected to vary from one design task to another.

In addition to the behavioral requirements there exist several specifications which the design parameters must fulfill. These are called side constraints and arise for reasons such as

a. analysis limitations
b. compatibility constraints
c. fabrication limitation

Technology

The method of analysis to be used in any synthesis program is a prerequisite to development of the synthesis capability. Existing literature contains methods of analysis which adequately predict the behavior of a substantial class of structural systems.

Criterion

In many important structural design areas the minimization of weight is important. It should be noted that a minimum weight basis for evaluating merit is probably the most readily stated and it is certainly of great importance in the design of flight vehicles.

The concepts of structural synthesis in no way necessitate the use of the weight function as the merit function. If another such measure, i.e., total cost, thermodynamic or aerodynamic performance is expressible mathematically, it may be used in place of the total weight with no conceptual changes in the synthesis process described herein.

SYMBOLS

A_{mn}	participation coefficient of assumed mode
a	x dimension of plate
b	y dimension of plate
b_L	lower bound on b_x
b_m	upper bound on b_x
b_x	spacing of stiffeners
C_a	shear buckling coefficient
D_1	bending stiffness - x direction
D_2	bending stiffness - y direction
D_3	torsional stiffness
D_p	p^{th} design parameter
E	modulus of elasticity
H	total height of stiffener plus skin
K_2	lower bound on t_s
K_3	lower bound on t_w

N_x	intensity of resultant normal force - x direction
N_{xy}	intensity of resultant shear force - x and y directions
N_y	intensity of resultant normal force - y direction
R_i	i^{th} random number
t_w	stiffener thickness
t_s	sheet thickness
w	z component of displacement
W	total weight of a waffle plate
Y	tensile yield stress
α_r	aspect ratio
μ	Poisson's ratio
ρ	weight density

FUNDAMENTALS OF STRUCTURAL SYNTHESIS

Basic Definitions

At the outset, certain parameters of any synthesis problem are set as design requirements. All those parameters which are not predetermined by the requirements are called design parameters. These independent variables are then determined by the synthesis program such that the merit function assumes the optimal value. Consider an n^{th} order space where the design parameters are plotted along the coordinate axes defining the space. This space will be referred to as a design parameter space. Note that it differs from the space frequently employed in optimization studies in that the merit function is not one of the coordinate axes. Instead contours of constant merit function are plotted in this space. Therefore, it is possible for the gradient to the merit function to assume a unique and distinct value for each point in the design parameter space. The coordinates of any point fix certain values to the independent design parameters, thus completely specifying the design of a structural system.

Also present in this space are behavioral constraint surfaces generated by the limitations on the structural behavior of the system. If a design point is on a behavioral constraint surface, the structure is on the verge of failure in one of the defined modes. This type of a design point is called a bounded point. Associated with each defined failure mode there exists a behavioral constraint surface for each load condition of the design load system. The collection of these surfaces which separates the acceptable region of the design space from the unacceptable region is called the composite constraint surface. This composite surface is continuous but, in general, the gradient to the composite surface is discontinuous at the junction of any two component surfaces. If any design point is above the composite surface in the sense that the design is adequate to sustain the loads, the design is conservative and this point is called an acceptable free point. If any design point is below the composite surface, the structure will not sustain the loads and the point is in the region of violation and the design is unacceptable. Note that it is possible for a design point to be bounded and unacceptable if it is on any constraint surface and below the composite surface. (See Fig. 2)

Behavior Functions

The behavior of the structure is tested or examined through the mechanism of a behavior function. A behavior function is a mathematical expression relating the coordinates of the proposed design and the design requirements to the behavior of the structure. If the behavior function assumes its limiting value the structure is on the verge of failure. If it is within its limits, the structure can successfully sustain the loads, while if the behavior function is beyond its limits the structure has failed. The most general matrix expression of the behavior functions for a single load condition is as follows:

$$\left\{ BF\ (D_p) \right\} = \left\{ \begin{matrix} g_1\ (D_p) \\ \vdots \\ g_n\ (D_p) \end{matrix} \right\}$$

If the values of the behavior functions are to be examined for more than one load condition, they can be displayed as columns of a rectangular array.

$$\left[BF\ (D_p) \right] = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{15} \\ \vdots & & & \vdots \\ g_{n1} & g_{n2} & \cdots & g_{n5} \end{bmatrix} \qquad (1)$$

If the following relationships holds

$$\left[L \right] \leq \left[BF\ (D_p) \right] \leq \left[U \right] \qquad (2)$$

where $\left[L \right]$ and $\left[U \right]$ are the lower and upper bounds, the design is said to be adequate, otherwise the design is unacceptable. Note that an upper bound of five load conditions was used in Eq. (1). This is arbitrarily selected as the upper bound for this study.

The quantity $\{D_p\}$ is a column vector of design parameters specifying the coordinates of the proposed point in the design parameter space.

Weight Function

The total weight of the structure, used as the sole criterion of design selection, can also be expressed as a function of the design parameters. In general, it is written as:

$$W = \sum_i \rho_i f_i (D_p) \tag{3}$$

The limitations on the design parameters may be expressed mathematically as follows:

$$\{D_L\} \leq \{D_p\} \leq \{D_U\} \tag{4}$$

General Problem

The mathematical statement of the synthesis of a structural system may now be summarized as follows:

<u>Given</u> ρ_i and the design load system $[N]$ as well as the design requirements $\{D_U\}$, $\{D_L\}$, $[L]$ and $[U]$.

<u>Find</u> $\{D_p\}$ such that

$$\{D_L\} \leq \{D_p\} \leq \{D_U\}$$

and

$$[L] \leq [BF(D_p)] \leq [U]$$

and

$$W = \sum_i \rho_i f_i (D_p)$$

assumes a minimum value.

Design Modification

The first step in any design problem is the establishment of an adequate trial design. Successful modification of the design can be accomplished by moving in the design parameter space, such that the merit function does not increase. For simplicity, the motion is restricted to straight lines and can be stated as follows:

$$\left\{ D_p^{(i+1)} \right\} = \left\{ D_p^{(i)} \right\} + \left\{ \psi \right\} t \qquad (5)$$

where

$\left\{ \psi \right\}$ are the direction cosines of the straight line of travel

t is the distance travelled

and the i superscript is the design cycle counter.

A major aspect of the development of methods of structural synthesis is the selection of proper directions and distances of travel in design parameter space. The following is a list of some of the available methods:

Directions

a. random - employ a random number generator to develop the direction cosines

b. semi-intelligent - orient a line in the design parameter space emanating from this current design to a predetermined point, e.g., point of equal weight or zero weight. This method was used in Reference 1, for the three bar truss problem.

c. intelligent - steepest descent.

Distances

a. arbitrary - fixed increment or a random increment

b. accelerated - select a fixed increment and move that distance. If the new point is acceptable double the distance and repeat this doubling until the design is in the region of violation. At this time halve the total distance of travel back to an already acceptable point. Place the design in permanent storage and restart from this new point with the original fixed increment. Use the doubling and halving scheme until the design point has converged to a constraint surface.

c. exact - solve for the distance to a neighboring constraint surface or a point on the same weight contour.

Various schemes of design modification can be generated by combining the above mentioned directions and distances of travel.

EXAMPLE PROBLEM

Symmetric Waffle Plate

As an example problem illustrating the application of the structural synthesis concept, consider a symmetric waffle plate subject to membrane loading. (See Figures 1 and 3). The waffle is symmetric in the sense that the stiffener spacing in the x direction equals the stiffener spacing in the y direction and the x stiffener thickness equals the y stiffener thickness.

The waffle plate is fabricated from a solid plate by first applying the appropriate protective coating and subjecting the plate to a chemically active etchant. This process, best known as chemical milling, has recently become feasible on a production basis. It has created the capability for producing stiffened panels with orthogonal and skewed sets of stiffeners integral with each other and integral with the back up sheet. It should be noted that all of the stiffeners are of the same depth, thus creating a flush inner surface. In the past, stiffened panels were fabricated by fastening stiffeners to the sheet, i.e., welding, riveting, etc. Fabrication problems led to some rather awkward configurations(joggles, clips, etc.) for nonparallel sets of stiffeners. The end result was that the design was difficult to fabricate and inefficient in terms of weight. Integral orthogonal stiffeners and sheets made it possible to realize significant weight savings.

The advance in fabrication capability due to chemical milling has created an interest in the analytical aspects of the problem. Because such structures were available for design applications, it became necessary to develop an analysis capable of accurately predicting the behavior of the structure. An analysis based on failure modes including gross instability, local buckling and yielding is presented in Appendix B.

The total weight of the structure is chosen as the sole criterion by means of which choices are made between various designs. It is fortuitous that a merit function, so significant to the prime users of such structures, is easily expressible mathematically. The concepts of structural synthesis in no way necessitate the use of the weight function as the merit function. If another measure, i.e., total cost, thermodynamic or aerodynamic performance, is expressible mathematically, it may be used in place of the total weight with no conceptual changes in the synthesis techniques described herein.

The design parameters selected for synthesis of symmetric waffle plate are: t_s, the sheet thickness; b_x, the stiffener spacing; and t_w, the stiffener thickness.

Symmetric Waffle Behavior Functions

As an example of a behavior function consider the material yield criterion as outlined in Appendix B. The waffle back-up sheet is on the verge of failure if the following condition exists

$$\frac{1}{H}\left\{\frac{\langle N_x^2 - N_x N_y + N_y^2 \rangle}{\left[\left(\frac{t_s}{H}\right) + \left(1 - \frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)\right]^2} + 3\left(\frac{N_{xy}}{t_s/H}\right)^2\right\}^{1/2} = Y \qquad \text{(B19)}$$

If the left-hand side is less than the value of Y, the design is adequate. If it is greater than Y, the back-up sheet has yielded. The left-hand side of Eq. (B19) is therefore already in the behavior function form. One alteration is made to facilitate computer calculation. The value of Y is taken to the left-hand side to give the following expression:

$$GY(D_p) = \frac{1}{YH}\left\{\frac{\langle N_x^2 - N_x N_y + N_y^2 \rangle}{\left[\left(\frac{t_s}{H}\right) + \left(1 - \frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)\right]^2} + 3\left(\frac{N_{xy}}{t_s/H}\right)^2\right\}^{1/2} \leq 1 \qquad (6)$$

Note that because of the nature of the function there is no lower limit. It was found that all of the behavior functions have the same basic characteristics

a. upper bound of unity
b. no finite lower bound
c. nonlinearly dependent upon the design parameters

The column matrix of behavior functions for a single load condition can now be written as follows:

$$\left\{ BF(D_p) \right\} = \left\{ \begin{array}{c} GY(D_p) \\ SX(D_p) \\ SY(D_p) \\ GBF(D_p) \\ LBX(D_p) \\ LBY(D_p) \\ LBP(D_p) \end{array} \right\} \leq \left\{ 1 \right\} \qquad (7)$$

where the elements of the behavior matrix represent the behavior functions associated with the following failure modes:

$GY(D_p)$	-	gross yield
$SX(D_p)$	-	stiffener (X direction) yield
$SY(D_p)$	-	stiffener (Y direction) yield
$GBF(D_p)$	-	gross plate buckling
$LBX(D_p)$	-	local stiffener buckling (X direction)
$LBY(D_p)$	-	local stiffener buckling (Y direction)
$LBP(D_p)$	-	local buckling of the back-up sheet

In order for the behavior of a design to be acceptable, the above criterion (7) must not be violated in any load condition. The inequality of equation (7) is defined as follows:

> Element by element the left-hand side of equation (7) must be less than or equal to unity for the equation to be satisfied. If any single element is greater than unity the inequality is not satisfied.

Symmetric Waffle Weight Function

The total weight of the symmetric waffle plate, used as the merit function, is written as follows:

$$W = ab\rho H \left[1 - (1 - \frac{t_s}{H}) (1 - \frac{t_w}{b_x})^2 \right] \qquad (8)$$

where t_s, t_w and b_x are the design parameters and a, b, ρ, H are defined in the Table of Symbols.

Design Parameter Limits

The column vector locating any design point in the design parameter space is defined as follows:

$$\left\{D_p\right\} = \left\{\begin{array}{c} t_s \\ b_x \\ t_w \end{array}\right\} \tag{9}$$

In general, there are three types of design parameter limits:

a. Fabrication limitations, e.g., lower bounds on sheet and stiffener thicknesses.

b. analysis limitations, e.g., upper bound on stiffener spacing.

c. compatibility constraints, e.g., stiffener spacing must be greater than the stiffener thickness.

The following are the matrix formulations of the upper and lower bounds. The letters a, b and c to the right of each element are the types of limits defined above:

$$\left\{D_U\right\} = \left\{\begin{array}{c} H \\ b_m \\ b_x \end{array}\right\} \begin{array}{l} b \\ b \\ c \end{array} \tag{10}$$

$$\left\{D_L\right\} = \left\{\begin{array}{c} K_2 \\ b_L \\ K_3 \end{array}\right\} \begin{array}{l} a \\ c \text{ and } a \\ a \end{array} \tag{11}$$

The bounds on the design parameters are employed as follows:

$$\left\{D_L\right\} \leq \left\{D_p\right\} \leq \left\{D_U\right\} \tag{12}$$

A more complete discussion of the design parameter bounds for symmetric waffle plate is presented at the end of Appendix B.

Brief Statement of the Problem

Now that the behavior functions and the merit function have been defined for the symmetric waffle, the mathematical statement of the example problem used in this study can be stated as follows:

> Given H, ρ and $[N]$, the design load system as well as the design requirements $\{D_U\}$ and $\{D_L\}$.
>
> Find $\{D_p\}$ such that

$$\{D_L\} \leq \{D_p\} \leq \{D_U\}$$

and

$$[BF(D_p)] \leq [1]$$

and

$$W = \rho abH \left[1 - (1 - \frac{t_s}{H})(1 - \frac{t_w}{b_x})^2 \right]$$

assumes a minimum value.

Design Modification

Three basic programs were written for the synthesis of waffle plates drawing on the techniques outlined previously. The first, Monte Carlo, was found to be inadequate to solve the problem, because of relative minima. The second, Compromise I, was inefficient in terms of computer time. The last, Compromise II was found to be both adequate and efficient. The following is a detailed description of all three programs.

Monte Carlo

As a first attempt, a modified Monte Carlo method was adopted for the design modification process. The process is as follows:

a. Propose a trial design and examine the behavior. If the design is adequate, proceed. If not, propose a new trial and repeat Step a.

b. Orient a unit vector from the current trial design to the point of absolute minimum weight. This point is at the intersection of the lower bounds of t_s and t_w and the upper bound on b_x.

c. Travel along this semi-intelligent direction until a constraint surface is encountered. Accelerated incremental motion is used to select the distance of travel.

d. Once the composite constraint surface is encountered use a random number generator to select the direction cosines of a new line of travel. The random direction must fulfill two requirements.

   1. must not enter the region of violation
   2. must not increase the merit function

e. Once the composite constraint surface is encountered using the random direction of travel a new random direction must be chosen. This process is repeated until convergence occurs and it is no longer possible to find an acceptable random direction.

Inherent in this method is the assumption that there are no relative minima in the design parameter space. It is readily seen that if there are no relative minima then convergence to a unique optimum can be accomplished regardless of the starting point. It was found that the assumption was not valid. Convergence to distinct optima did occur and the final output design was greatly influenced by the initial trial. A case which exhibited two distinct optima was examined in greater detail. A straight line was drawn between the two optimal points as shown in Figure 4. The behavior functions were examined at fixed increments along the line. It was found that the gross behavior function increased from the bounded value of unity as the increment moved the test point away from the first optimum and then gradually decreased to unity as the other optimum point was approached. This test showed that pockets exist in the gross behavior function and consequently in the composite constraint surface. Therefore, it was concluded that the modified Monte Carlo program was inadequate to synthesize waffle plates. It became necessary to develop a method of piercing the constraint surface to examine the possibility of relative minima pockets.

Compromise I

Compromise I was the first attempt at solving the synthesis problem recognizing the existence of relative minima. The method or program was named Compromise because it was the merger of two methods of direction selection and two methods of distance determination. The program employs the highly intelligent steepest descent direction of travel with an incremental distance of travel whenever the current trial design is free and acceptable. Steepest descent motion proceeds until a constraint surface is encountered. When the current design is bounded to a constraint surface a random number generator is used to select a direction. The intersections of this random direction with the current weight contour are found and tested as trial designs. If one of these "side-step" points is acceptable and free it is placed in permanent storage and steepest descent motion proceeds until a constraint surface is encountered.

Steepest Descent

The direction of travel in steepest descent is found by calculating the components of the gradient to the weight surface, at each trial design point.

$$\frac{\partial W}{\partial t_s} = \left(1 - \frac{t_w}{b_x}\right)^2 \rho ab \tag{13}$$

$$\frac{\partial W}{\partial b_x} = -\frac{2t_w}{b_x^2}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_w}{b_x}\right)\rho ab\, H \tag{14}$$

$$\frac{\partial W}{\partial t_w} = \frac{2}{b_x}\left(1 - \frac{t_s}{H}\right)\left(1 - \frac{t_w}{b_x}\right)\rho ab\, H \tag{15}$$

The above components of the gradient to the weight surface are normalized as follows and are used as the direction cosines

ϕ :

$$\phi_{t_s} = \frac{\frac{\partial W}{\partial t_s}}{\sqrt{\left(\frac{\partial W}{\partial t_s}\right)^2 + \left(\frac{\partial W}{\partial b_x}\right)^2 + \left(\frac{\partial W}{\partial t_w}\right)^2}} \tag{16}$$

$$\phi_{b_x} = \frac{\frac{\partial W}{\partial b_x}}{\sqrt{\left(\frac{\partial W}{\partial t_s}\right)^2 + \left(\frac{\partial W}{\partial b_x}\right)^2 + \left(\frac{\partial W}{\partial t_w}\right)^2}} \tag{17}$$

$$\phi_{t_w} = \frac{\frac{\partial W}{\partial t_s}}{\sqrt{\left(\frac{\partial W}{\partial t_s}\right)^2 + \left(\frac{\partial W}{\partial b_x}\right)^2 + \left(\frac{\partial W}{\partial t_w}\right)^2}} \tag{18}$$

An increment of 0.01 was used for the steepest descent distance of travel. As the design point approaches a constraint surface it is possible that this increment is too large. If this is the case, the distance of travel is found by systematically halving the current distance of travel until the design point lies on the constraint surface, with a fixed tolerance ϵ. Once the design point is on a constraint surface, it is generally impossible to steep descend without piercing through the constraint surface. A side-step or alternate step is then sought such that the merit function remains constant.

Alternate Step

Figure 5 is a sketch of a planar slice through a design parameter space. The plane is determined by a random line in the plane of $t_w - b_x$ and the current design point. Two random numbers are needed to determine the random line in the base plane and a third random member is used to generate the direction of travel in the random plane. The design modification using the random direction can be written as follows:

$$t_s^{(i+1)} = t_s^{(i)} + \psi_1 \Delta$$

$$b_x^{(i+1)} = b_x^{(i)} + \psi_2 \Delta$$

$$t_w^{(i+1)} = t_w^{(i)} + \psi_3 \Delta$$

where Δ is the distance to another point on the same weight contour.

Equate the weight of the i^{th} design to the weight of the $(i+1)^{th}$ design

$$W^{(i)} = W^{(i+1)}$$

cancelling common terms the result is as follows

$$(1 - \frac{t_s}{H})(1 - \frac{t_w}{b_x})^2 = (1 - \frac{t_s + \psi_1 \Delta}{H})(1 - \frac{t_w + \psi_3 \Delta}{b_x + \psi_2 \Delta})^2 \qquad (19)$$

Solve for the polynomial in Δ.

$$b_x^2 \frac{\psi_1}{H} (\psi_2 - \psi_3)^2 \Delta^3$$

$$+ \Big[2b_x^2 \frac{\psi_1}{H} (b_x - t_w)(\psi_2 - \psi_3) - b_x^2 (1 - \frac{t_s}{H})(\psi_2 - \psi_3)^2$$

$$+ \psi_2^2 (1 - \frac{t_s}{H})(b_x - t_w)^2 \Big] \Delta^2 + \Big[b_x^2 \frac{\psi_1}{H} (b_x - t_w)^2$$

$$- 2b_x^2 (1 - \frac{t_s}{H})(b_x - t_w)(\psi_2 - \psi_3)$$

$$+ 2b_x \psi_2 (1 - \frac{t_s}{H})(b_x - t_w)^2 \Big] \Delta = 0 \qquad (20)$$

There is a common factor of Δ, indicating a zero root. This is reasonable because the current design point is on the same weight contour. Notice from Figure 5 that it is possible to construct a line which intersects the weight contour at only one point, i.e., the current design point. This particular line will yield a pair of complex roots to the polynomial in Δ , Eq. (20). If such a case is generated, the computer program is written to immediately reject that direction and generate three new random direction cosines. It should be mentioned that the random consines are not independent in the sense that they are generated by normalizing three random numbers.

$$\psi_i = \frac{R_i}{\sqrt{\sum_i R_i^2}} \tag{21}$$

where i = 1, 2, 3
and R_i are random numbers from minus to plus unity.

The zero root of the cubic equation (20) is dropped and the computer program treats the equation as a general quadratic. If complex roots are generated they too are dropped and another set of random numbers are generated. If real roots are found as the solution, they are then employed to select an alternate step design point. If the resulting design is a free acceptable point, the program returns to the steepest descent mode of travel until a constraint surface is encountered again. The process is repeated until convergence to an optimum occurs. Since the method either reduces the weight or holds it constant, it is not possible for the solution to diverge away from the desired optimum.

Proof of an Optimum

The major shortcoming of this technique, as with any search technique, is the proof of the optimum. Once the program seems to have converged to a minimum, the question asked is as follows:

> Is this the absolute minimum or is it "locked-in", so to speak, at a relative minimum? The same question can be stated with a negative slant. Is it possible to find a direction, random or otherwise, which will lead to a design point of equal or lower weight, above the composite constraint surface?

Since there is no mathematical means of answering the positive question, the latter must be used. A rather laborious but conclusive means of answering the negative question would be to examine all admissible designs of weight equal to the weight of the apparent optimum. For the case of three independent design parameters, there is a second order infinite set of design possibilities.

Since the behavior functions and the weight function are well behaved, it is only necessary to examine designs at a finite increment, for the fixed weight. If the current design is the only design in its weight contour within the design parameter bounds, sufficient to sustain the loads, then the current design must be regarded as the true optimum.

This weight grid method was used for some of the preliminary work. After the computer program found an optimum from two starting points, the optimum weight was used to generate a weight grid. In every case, the weight grid showed that the apparent optimum was truly the absolute minimum.

Instead of examining a gridwork of admissible designs, a random selection of designs at the current weight can be tested. The more designs tested, the higher the degree of confidence that the apparent optimum is the true optimum. This test of the optimum can be accomplished by permitting the computer to search for an alternate step after it has appeared to have converged.

Another method of verifying the optimum, although it is not conclusive, is to run the synthesis from two distinct trial design points. If the synthesis paths are different but the final optimum is the same, within the same computational tolerance, a degree of certainty can be achieved. The method used for this study is a combination of the latter two methods. Each design case is synthesized from two distinct trial points and once the proposed optimum is reached the computer is permitted to run and randomly examine the field of admissible designs of optimal weight. In this way, a degree of confidence is developed in the optimum design.

Compromise II

Compromise II is an advanced and more intelligent version of its predecessor, Compromise I. It was found that Compromise I consumed computer time in searching through the random directions to find a line which would yield another point on the same weight contour. Compromise II reduces the degree of randomness and examines only those directions which will yield an alternate step within the design parameter bounds. Therefore, it is more selective in its directions and consequently more intelligent.

The random number generator is again employed to generate a random line in the base plane $b_x - t_w$. This line is used in conjunction with the current design point to generate the random plane as was done in Compromise I. Figure 6 shows that if an alternate step design exists within the design parameter limits, it must be along the weight contour and within the following bounds:

a. the tangent to the weight surface
b. the design parameter bounds
c. the chords defined by the current design point and the intersection of the weight contour and the design parameter limits.

Notice that the existence of this design is independent of the behavior. First, an alternate step design point within the design parameter bounds must be found and then its behavior examined. If the behavior is adequate, the alternate step point is accepted and re-design is carried out along the line of steepest descent as described in the steepest descent section for Compromise I. The only difference between Compromise I and Compromise II is that in the latter the selection of a direction for seeking an alternate step is substantially more efficient.

Alternate Step

A random number generator is employed to select a random plane parallel to the t_s axis, as was done in Compromise I. Figure 7 shows a view of the design parameter space looking down the t_s axis. Notice that there are three distinct sets of random lines in the base plane. That is, the random plane generated by the random line and the current design point, can intersect the design parameter bounds in three combinations.

Case 1 - Upper bound on b_x and compatability bound ($b_x > t_w$)

Case 2 - Lower bound on t_w and compatability bound ($b_x > t_w$)

Case 3 - Upper bound on b_x and lower bound on t_w

The random line in the plane of $b_x - t_w$ can be written mathematically as follows:

$$Bb_x + Ct_w = D_c \qquad (22)$$

where

$$B = \frac{R_1}{\sqrt{R_1^2 + R_2^2}}$$

$$C = \frac{R_2}{\sqrt{R_1^2 + R_2^2}}$$

R_i are random numbers

and

D_c is determined by evaluating (22) at the current design point

The components of the gradient to the weight surface are given by equations (13) (14) and (15) and are used to write an expression for the plane tangent to the weight surface at the current design

$$\left[\frac{1}{H} \left(1 - \frac{t_w}{b_x}\right)^2 \right]_c t_s - \left[2 \frac{t_w}{b_x^2} \left(1 - \frac{t_s}{H}\right) \left(1 - \frac{t_w}{b_x}\right) \right]_c b_x + \left[2 \frac{1}{b_x} \left(1 - \frac{t_s}{H}\right) \left(1 - \frac{t_w}{b_x}\right) \right]_c t_w = K_c \tag{23}$$

where the subscript c indicates that the quantity within the brackets must be evaluated at the current design point. K_c must also be determined at the current design point.

Figure 8 is a view of a random slice through a design parameter space showing the weight surface and a hypothetical composite constraint surface. The plane of the paper is the plane determined by the random line in the $b_x - t_w$ plane and the current design point. The type or case of the random plane must be ascertained in order to determine the appropriate design parameter bounds. The following is a scheme for classifying the types of random lines:

a. generate the random numbers R_1 and R_2 and solve for B, C, and D_c

b. set t_w to zero and examine the resulting value of b_x. If b_x is less than its lower bound or higher than its upper bound, the random line is of Case 1.

c. if the above test fails the random line must be of Case 2 or 3. Set b_x to zero and examine the resulting t_w. If t_w is positive the random line is of Case 2. If negative, it is Case 3.

Having once determined the case of the random line used to generate the random plane, it is possible to solve for the coordinates of the intersection of

a. the random plane, the tangent to the weight surface and the appropriate design parameter bounds.

b. the random plane, the contour of the current weight and the appropriate design parameter bounds.

These coordinates are found by the simultaneous solution of the expressions for the above mentioned surface. The points are subsequently used to solve for an alternate step vector in the design parameter space.

In Figure 9, the tangent intersections are labeled (2) and (4) while the weight contour intersections are labeled (1) and (3). The current design point is labeled (5). Notice that an alternate step direction of travel emanating from the current design and intersecting the weight contour within the design parameter bounds must pass between points (1) and (2) or points (3) and (4). Compromise II selects alternate step directions of travel by first solving for a vector from point (5) to point (1). It adds to this base vector a fraction of the vector from point (1) to point (2). A similar vector is generated by using points (3) and (4). It then normalizes the resulting vector and employs the root finder polynomial (20). Only one root to the polynomial has any significance, and that is the distance to a point on the same weight contour along the line generated. The major asset to this method is that the program wastes no time in generating and testing designs which have no real physical significance. The program is more efficient simply because it is more intelligent in its selection of an alternate step direction of travel.

In order to solve for the coordinates of the intersections, it is necessary to solve equations (22), (23), (8) and the appropriate design parameter bounds, simultaneously. There are three design parameter bounds, ($t_s = 0$, $t_w = 0$, $b_x = (b_x)_{max}$) and the necessary sets of coordinates are as follows:

(a) Point 1

$$b_{x_1} = (b_x)_{max}$$

$$t_{w_1} = \frac{D_c - Bb_{x_1}}{C}$$

$$t_{s_1} = H\left[1 - \frac{\left(1 - \frac{W_c}{ab\rho H}\right)}{\left(1 - \frac{t_{w_1}}{b_{x_1}}\right)^2}\right]$$

Point 2

$$b_{x_2} = (b_x)_{max}$$

$$t_{w_2} = \frac{D_c - Bb_{x_2}}{C}$$

$$t_{s_2} = \frac{K_c - \left(\frac{\partial W}{\partial b_x}\right)_c b_{x_2} - \left(\frac{\partial W}{\partial t_w}\right)_c t_{w_2}}{\left(\frac{\partial W}{\partial t_s}\right)_c}$$

(b) Point 3 $t_{s_3} = 0$

$$b_{x_3} = \frac{D_c}{B + \left[1 - \left(1 - \frac{W_c}{ab\rho H}\right)^{1/2}\right] C}$$

$$t_{w_3} = \frac{D_c \left[1 - \left(1 - \frac{W_c}{ab\rho H}\right)^{1/2}\right]}{B + \left[1 - \left(1 - \frac{W_c}{ab\rho H}\right)^{1/2}\right] C}$$

Point 4 $t_{s_4} = 0$

$$b_{x_4} = \frac{1}{B}\left[D_c - C\left(\frac{\left(\frac{\partial W}{\partial b_x}\right)_c D_c - B K_c}{\left(\frac{\partial W}{\partial b_x}\right)_c C - \left(\frac{\partial W}{\partial t_w}\right)_c B}\right)\right]$$

$$t_{w_4} = \frac{\left(\frac{\partial W}{\partial b_x}\right)_c D_c - B K_c}{\left(\frac{\partial W}{\partial b_x}\right)_c C - \left(\frac{\partial W}{\partial t_w}\right)_c B}$$

(c) Point 1 $t_{w_1} = 0$

$$b_{x_1} = \frac{D_c}{B}$$

$$t_{s_1} = \frac{W_c}{ab\rho}$$

Point 2 $t_{w_2} = 0$

$$b_{x_2} = \frac{D_c}{B}$$

$$t_{s_2} = \frac{1}{\left(\frac{\partial W}{\partial t_s}\right)_c}\left[K_c - \frac{\left(\frac{\partial W}{\partial b_x}\right)_c D_c}{B}\right]$$

(d) Point 3 $b_{x_3} = (b_x)_{max}$

$$t_{w_3} = \frac{D_c - Bb_{x_3}}{C}$$

$$t_{s_3} = H\left[1 - \frac{\left(1 - \frac{W_c}{ab\rho H}\right)}{\left(1 - \frac{t_{w_3}}{b_{x_3}}\right)^2}\right]$$

Point 4 $b_{x_4} = (b_x)_{max}$

$$t_{w_4} = \frac{D_c - Bb_{x_4}}{C}$$

$$t_{s_4} = \frac{K_c - \left(\frac{\partial W}{\partial b_x}\right)_c b_{x_4} - \left(\frac{\partial W}{\partial t_w}\right)_c t_{w_4}}{\left(\frac{\partial W}{\partial t_s}\right)_c}$$

The coordinates of the intersections for the different cases of random lines are as follows:

Case 1 - coordinate sets (a) and (b)
Case 2 - coordinate sets (c) and (b)
Case 3 - coordinate sets (c) and (d)

The coordinates are used to generate a vector emanating from the currents design to a region bounded by the tangent and the chord. Figure 8 is another sketch of an arbitrary slice through the design parameter space showing the weight contour and a hypothetical composite constraint surface. Knowing the case of the random line, it is possible to determine the coordinates of Points a, b, c and d. The alternate step direction of travel is generated as follows:

$$\bar{r}_1 = (\bar{a} - \bar{p}) + \gamma\,(\bar{b} - \bar{a}) \tag{24a}$$

or

$$\bar{r}_2 = (\bar{c} - \bar{p}) + \gamma\,(\bar{d} - \bar{c}) \tag{24b}$$

and

$$\bar{\psi}_1 = \frac{\bar{r}_1}{|\bar{r}_1|} \tag{25a}$$

$$\bar{\psi}_2 = \frac{\bar{r}_2}{|\bar{r}_2|} \tag{25b}$$

when the bars indicate vectors and γ is a positive fraction less than unity. It is possible to examine several points along the weight contour between p and a or p and c by using different values for γ. If γ is incremented from zero to unity the alternate step design point moves along the contour from a to p. It is readily seen that is γ is incremented from unity, the first alternate step tested will be acceptable unless the current point is a relative minimum. The synthesis path will tend to creep down the side of the constraint surface. On the other hand, if γ is incremented upward from zero the synthesis path will tend to oscillate across the acceptable region. Both of these incrementation schemes lead to a slow convergence. The incrementation of γ should lead to an alternate step design point which is not near any constraint surface such that a maximum distance of travel can be achieved in steepest descent. The following is the sequence of values of γ used in this work.

0.5000
0.7500
0.2500
0.6250
0.3750
0.8750
0.1250
0.0625
0.9375
0.03125

The sequence is arranged so that the alternate step direction will tend to direct the path toward the center of the acceptable region. If the current design point is in a large pocket the tendency is for the alternate step to remain in that same pocket. On the other hand, if the design point is in a small pocket, it is possible for the synthesis path to leave that pocket if a more dominant relative minimum exists. Figures 10 and 11 show flow charts for the analysis and the Compromise II synthesis program respectively.

In summary, the method will select alternate step directions of travel which lie in a random plane between an upper bound (the tangent) and a lower bound (the chord). If the design point is free, the synthesis path is along an incremental steepest descent. Compromise II was found to be considerably faster than Compromise I. Identical problems were solved using both programs to compare running time. While Compromise I needed approximately 90 minutes to solve a given problem, Compromise II converged to the optimum in approximately 30 minutes. All programs were written using the Runcible compiler and the Burroughs 220 Machine.

The problem of proving an optimum is still present in Compromise II. The method used is the same as that of Compromise I.

NUMERICAL RESULTS

Contained within this section are various numerical examples of waffle synthesis examined as verification of the capability. The output from the Monte Carlo program is presented only to show its inadequacy in synthesizing waffle plates. because of the existence of relative minima. Compromise I, although capable of solving the relative minima problem, was not used as a production program; consequently, no output is presented. All remaining numerical results were found using Compromise II.

Six sets of input data and optimal designs are presented in this chapter. The problems are labeled

case i - j

where i is the number of load conditions and j is the case member.

Two synthesis paths are presented to show the design evolution from the initial trial, through relative minima pockets, and on to the final design.

It is important to remember that the only bound on the value of a behavior function is an upper bound of unity. The values of the behavior functions, significantly different from unity, are important only in the sense that the structure is not on the verge of failure in that particular mode. That is to say, the only behavior function bounds which have any bearing on the optimum design are those which have values near unity. An * on numeric values is used to indicate those bounds which constrain the optimum design. The design parameter symbols marked with * are the optimal values.

All numerical work was carried out on Case's Burroughs 220 Digital Computer using the Runcible compiler language.

INPUT DATA FOR CASES 1-1, 1-2 and 1-3

COMMON DATA

$a = 40''$ $\quad$ $b = 30''$

$E = 10.5 \times 10^3$ ksi $\quad$ $\rho = 0.101$ lbs/cu.in.

$Y = 72.0$ ksi $\quad$ $\mu = 0.32$

$$\{N\} = \begin{Bmatrix} -0.30 \\ -0.40 \\ +0.20 \end{Bmatrix} \qquad \epsilon = 0.0001$$

$$\{D_U\} = \begin{Bmatrix} H \\ 6.00 \\ b_x \end{Bmatrix} \qquad \{D_L\} = \begin{Bmatrix} 0.005 \\ b_L \\ 0.010 \end{Bmatrix}$$

Total Depth (H) for each case

	1-1	1-2	1-3
H	0.4''	0.6''	0.8''

TRIAL DESIGNS

	Case 1-1 Monte Carlo	Case 1-1 Comp II	Case 1-2 Comp II	Case 1-3 Comp II
Point A				
t_s	0.300	0.300	0.300	0.010
b_x	5.000	5.000	5.000	2.100
t_w	4.000	4.000	4.000	1.950
Point B				
t_s	0.360		0.300	0.300
b_x	5.000		2.000	2.000
t_w	0.250		1.950	1.950

FINAL OUTPUT FOR CASES 1-1, 1-2, 1-3

	Case 1 - 1			1 - 2	1 - 3
	Monte Carlo			Compromise II	
	Point A	Point B			
t_s^*	0.2803	0.0838	0.2803	0.0397	0.0448
b_x^*	4.3734	5.2514	4.3735	2.5339	2.7709
t_w^*	0.0101*	2.2969	0.0101*	0.2358	0.0848
W^*	34.0346	36.3488	34.0346	16.8585	10.9423
	0.0248	0.0617	0.0248	0.1330	0.1304
	0.0149	0.0188	0.0149	0.0454	0.0614
	0.0198	0.0250	0.0198	0.0605	0.0818
BF^*	0.9996*	1.0000*	0.9996*	1.0000*	0.9961*
	0.0364	0.0000	0.0364	0.0040	0.0723
	0.0485	0.0000	0.0485	0.0053	0.0964
	0.0157	0.1029	0.0157	0.7010	0.9969*

INPUT DATA FOR CASES 3-1, 3-2 and 3-3

COMMON DATA

$a = 70''$ $\qquad$ $b = 50''$

$H = 0.5''$ $\qquad$ $\epsilon = 0.0001$

$$[N] = \begin{bmatrix} -0.30 & +0.50 & -1.00 \\ -0.30 & -0.60 & 0.00 \\ 0.00 & -1.00 & -0.40 \end{bmatrix}$$

$$\{D_U\} = \begin{Bmatrix} H \\ 6.00 \\ b_x \end{Bmatrix} \qquad \{D_L\} = \begin{Bmatrix} 0.005 \\ b_L \\ 0.010 \end{Bmatrix}$$

MATERIAL PROPERTIES FOR EACH CASE

	Case 3-1	Case 3-2	Case 3-3
E (ksi)	10.5×10^3	30.0×10^3	16×10^3
Y (ksi)	72	150.0	120.0
ρ (lbs/cu.in)	0.101	0.276	0.160
μ	0.32	0.283	0.290

TRIAL DESIGN

Point A	Point B
$t_s = 0.400$	$t_s = 0.400$
$b_x = 5.000$	$b_x = 2.000$
$t_w = 4.000$	$t_w = 1.950$

CASE 3-1

PROGRAM - COMPROMISE II

FINAL OUTPUT

POINT A

$$t_s^* = 0.3942 \qquad t_w^* = 0.0108^*$$

$$b_x^* = 3.1821 \qquad W^* = 139.6039$$

$$[BF] = \begin{bmatrix} 0.0106 & 0.0697 & 0.0428 \\ 0.0106 & 0.0176 & 0.0352 \\ 0.0106 & 0.0211 & 0.0000 \\ 0.8408 & 0.9701 & 0.9997^* \\ 0.0176 & -0.0294 & 0.0588 \\ 0.0176 & 0.0353 & 0.0000 \\ 0.0026 & 0.0003 & 0.0043 \end{bmatrix}$$

POINT B

$$t_s^* = 0.3941 \qquad t_w^* = 0.0104^*$$

$$b_x^* = 2.8271 \qquad W^* = 139.5998$$

$$[BF] = \begin{bmatrix} 0.0106 & 0.0697 & 0.0428 \\ 0.0106 & 0.0176 & 0.0352 \\ 0.0106 & 0.0211 & 0.0000 \\ 0.8411 & 0.9706 & 1.0000^* \\ 0.0192 & -0.0320 & 0.0640 \\ 0.0192 & 0.0384 & 0.0000 \\ 0.0020 & 0.0003 & 0.0034 \end{bmatrix}$$

CASE 3-2

PROGRAM - COMPROMISE II

FINAL OUTPUT

POINT A

$t_s^* = 0.0520 \qquad t_w^* = 0.6421$

$b_x^* = 3.7051 \qquad W^* = 187.1947$

$$[BF] = \begin{bmatrix} 0.0154 & 0.2276 & 0.1027 \\ 0.0154 & 0.0257 & 0.0514 \\ 0.0154 & 0.0309 & 0.0000 \\ 0.8432 & 0.9754 & 1.0000^* \\ 0.0000 & -0.0002 & 0.0003 \\ 0.0000 & +0.0002 & 0.0000 \\ 0.1500 & 0.0963 & 0.2614 \end{bmatrix}$$

POINT B

$t_s^* = 0.0495 \qquad t_w^* = 0.6459$

$b_x^* = 3.675 \qquad W^* = 187.3800$

$$[BF] = \begin{bmatrix} 0.0155 & 0.2382 & 0.1066 \\ 0.0155 & 0.0259 & 0.0518 \\ 0.0155 & 0.0311 & 0.0000 \\ 0.8432 & 0.9575 & 1.0000^* \\ 0.0000 & -0.0002 & 0.0003 \\ 0.0000 & 0.0002 & 0.0000 \\ 0.1624 & 0.1178 & 0.2852 \end{bmatrix}$$

CASE 3-3 PROGRAM - COMPROMISE II

FINAL OUTPUT

POINT A

$t_s^* = 0.3447 \qquad t_w^* = 0.0099^*$

$b_x^* = 4.0717 \qquad W^* = 193.4650$

$$[BF] = \begin{bmatrix} 0.0072 & 0.0478 & 0.0294 \\ 0.0072 & 0.0121 & 0.0241 \\ 0.0072 & 0.0145 & 0.0000 \\ 0.8408 & 0.9701 & 0.9996^* \\ 0.0348 & -0.0579 & 0.1159 \\ 0.0348 & 0.0695 & 0.0000 \\ 0.0042 & 0.0007 & 0.0070 \end{bmatrix}$$

POINT B

$t_s^* = 0.3445 \qquad t_w^* = 0.0112$

$b_x^* = 2.9069 \qquad W^* = 193.5714$

$$[BF] = \begin{bmatrix} 0.0072 & 0.0478 & 0.0294 \\ 0.0072 & 0.0121 & 0.0242 \\ 0.0072 & 0.0145 & 0.0000 \\ 0.8408 & 0.9701 & 0.9997^* \\ 0.0272 & -0.0453 & 0.0905 \\ 0.0272 & 0.0543 & 0.0000 \\ 0.0021 & 0.0004 & 0.0036 \end{bmatrix}$$

TABLE 1

SYNTHESIS PATH

CASE 1-1 PROGRAM - COMPROMISE II

Cycle	t_s	b_x	t_w	W
1	0.3000	5.0000	4.0000	47.9952
2	0.0334	5.0872	3.8886	46.0138
3	0.1722	5.3338	3.7397	46.0138
4	0.0348	5.3679	3.6905	44.1581
5	0.1149	5.6081	3.6247	44.1581
6	0.0361	5.6243	3.5995	42.7630
7	0.2142	4.9150	2.4388	42.7630
8	0.0531	4.9316	2.4051	37.4435
9	0.1221	5.0286	2.1502	37.4435
10	0.0997	5.0307	2.1453	36.4125
11	0.0978	5.2417	2.2331	36.4125
12	0.0782	5.1701	2.2943	36.4125
13	0.0779	5.1702	2.2942	36.4004
14	0.0929	5.2392	2.2547	36.4004
15	0.0919	5.2393	2.2545	36.3626
16	0.2768	5.2711	0.5234	36.3626
17	0.2669	5.2712	0.5229	35.3855
18	0.2794	3.3337	0.1785	35.3855
19	0.2736	3.3337	0.1781	34.7558
20	0.2802	3.7624	0.1045	34.7558
21	0.2771	3.7624	0.1043	34.3970
22	0.2804	4.2093	0.0595	34.3970
23	0.2788	4.2093	0.0594	34.2034
24	0.2805	4.3460	0.0310	34.2034
25	0.2796	4.3460	0.0310	34.1006
26	0.2805	4.3711	0.0156	34.1006
27	0.2801	4.3711	0.0156	34.0535
28	0.2803	4.3735	0.0117	34.0535
29	0.2802	4.3735	0.0117	34.0346
30	0.2803	4.3735	0.0101	34.0346

TABLE 2

SYNTHESIS PATH

CASE 3-3 PROGRAM - COMPROMISE II

Cycle	t_s	b_x	t_w	W
1	0.4000	5.0000	4.0000	277.7600
2	0.0500	5.1283	3.8343	263.9557
3	0.2113	5.3778	3.6836	263.9557
4	0.0519	5.4227	3.6171	252.1764
5	0.1384	5.6508	3.5546	252.1764
6	0.0534	5.6709	3.5224	244.1024
7	0.2675	4.9569	2.3540	244.1024
8	0.0525	4.9825	2.2994	207.3220
9	0.1377	5.0812	2.0402	207.3220
10	0.0911	5.0858	2.0285	197.2403
11	0.1097	5.1611	1.9854	197.2403
12	0.1094	5.1612	1.9853	197.1718
13	0.0980	5.1603	2.0302	197.1718
14	0.0971	5.1604	2.0300	196.9714
15	0.1018	5.2467	2.0453	196.9714
16	0.3383	4.1035	0.1746	196.9714
17	0.3374	4.1035	0.1746	196.5287
18	0.3442	4.1355	0.0906	196.5287
19	0.3413	4.1355	0.0903	194.9753
20	0.3447	4.0262	0.0449	194.9753
21	0.3432	4.0262	0.0447	194.1170
22	0.3449	4.0520	0.0227	194.1170
23	0.3441	4.0520	0.0226	193.6832
24	0.3450	4.0716	0.0114	193.6832
25	0.3446	4.0716	0.0114	193.4650
26	0.3447	4.0717	0.0099	193.4650

RESULTS AND DISCUSSION

This section is subdivided into two major sections: the examination of the results to show the characteristics of synthesis and the discussion of results in conjunction with certain preconceived notions of waffle design.

Two basic waffle plates are studied in detail. The first set is subjected to a single load condition and the design requirements are varied by changing the total depth of the structure. The second set of waffle plates is subjected to three load conditions and the design requirements are varied by changing the material properties. This second set of design problems shows how a structural synthesis capability may be used as a scientific aid in the selection of a material to do a specific job. The materials selected for the study are only typical alloys of aluminum, titanium and steel.

Synthesis Characteristics

Monte Carlo

Case 1-1 was synthesized using the first computer program Monte Carlo. It was found that the starting point had a significant influence on the final optimum. For example, when

$$t_s = 0.3000$$
$$b_x = 5.0000$$
$$t_w = 4.0000$$

was used as the starting point, the final output was:

Point A
$$t_s^* = 0.2803$$
$$b_x^* = 4.3734$$
$$t_w^* = 0.0101$$
$$W^* = 34.0346$$

and when

$$t_s = 0.3600$$
$$b_x = 5.0000$$
$$t_w = 0.2500$$

was used as the initial trial, the resulting output was

Point B

$$t_s^* = 0.0838$$
$$b_x^* = 5.2514$$
$$t_w^* = 2.2969$$
$$W^* = 36.3488$$

Notice that the two resulting designs, although different in weight by only 6 percent, are radically different in configuration. This is the first indication of the possibility of a relative minimum. Several other starting points were tried but each time the final output was either Point A or B. A highly specialized computer program, which examined the designs along a straight line between Points A and B, was then written. It was found that this trace in the design parameter space went under the composite constraint surface into the region of violation and more important re-entered the acceptable region. This indicates that there are relative minima pockets in the composite constraint surface when plotted in the design parameter space. It was observed that these two relative minima were both bounded by the gross buckling constraint surface. In fact, the whole region of the composite surface between the two relative minima could be attributed to the gross buckling constraint. Therefore, the relative minima are not necessarily created by union of the individual behavioral constraints into the composite surface but can be generated by an individual surface. The relative minima in the individual constraint surfaces are due to the polynomial nature of the analysis expressions.

A close examination of the expressions for the flexural and torsional rigidities and the sub-critical buckling expressions shows that the gross buckling behavior is not dependent upon the specific values of t_w and b_x but only the ratio t_w/b_x. The expression for a constant weight surface also has this characteristic. Therefore, it is possible to map the gross buckling constraint surface for a constant weight by a simple incremental variation to t_w/b_x. A second specialized computer program was written employing these characteristics. The procedure is as follows:

a. select a weight for a design problem (e.g., Case 1-1 use the higher of the two relative minima)

b. calculate the design for a maximum t_w/b_x ratio, i.e., minimum t_s, and examine the behavior of this design

c. reduce t_w/b_x by some fixed increment examining each design until t_w/b_x reaches a minimum value.

The output from this program indicates relative minima pockets even for a constant weight. Therefore, the relative minima pockets also exist in the corresponding weight space. The cubic nature of the gross buckling behavior function indicates there is a maximum of three values of sheet thickness for a fixed weight, thus indicating the relative minima are possible within a single constraint surface.

There are several other characteristics which can be studied for single load condition cases

a. the thin sheet relative minimum design may be governed by a single constraint surface. It is not necessarily true that the optimum design be at the intersection of two or more constraint surfaces.

b. if a second constraint surface is active in the thin sheet relative minimum pocket, it will be the local buckling of an individual panel; or the lower bound on the sheet thickness.

c. the thick sheet relative minima design on the other hand will, in general, lie at the intersection of two constraints:

1. gross buckling

and

2. local stiffener buckling or stiffener thickness lower bound.

Although it is not obvious, it is not possible at the outset to determine or predict which of the two distinct relative minima will be the absolute minima. The optimum design for Case 1-1 is a thick sheet design bounded by gross buckling and stiffener thickness lower bound. In this case the true optimum is most likely a plane sheet without stiffeners.

Case 1-2, on the other hand, has several relative minima within the thin sheet design pocket. The Monte Carlo program found the relative minima by starting from different initial trial points. These sub-relative minima pockets are due to the polynomial nature of the expressions for the local buckling behavior of the sheet.

Because of the existence of both the relative and the sub-relative minima the Monte Carlo program is inadequate to synthesize waffle plates. Since the final output is so heavily dependent upon the initial trial, it is impractical to use the Monte Carlo program to find all relative minima and choose the optimum. Therefore, it was necessary to develop a scheme of hopping from pocket to pocket, or piercing the constraint surfaces to examine hidden points.

Compromise I

Compromise I, described previously, has the capability of solving the relative minima problem. Because of the method of selecting alternate step directions of travel the running times were rather high. For example, a typical single load condition case required approximately 90 minutes of computer time. Compromise II reduced the running time by a factor of three and consequently Compromise I was not used as a production program.

Compromise II

The bulk of the production results was generated with the Compromise II waffle synthesis program. Case 1-1 synthesized with the Monte Carlo program was repeated using the Compromise II and converged to the thick sheet design, the lower of the two relative minima found with the Monte Carlo program.

The results substantially support the argument that Compromise II will converge to the absolute minimum within a field of relative minima.

The optimum design for Case 1-3 is a thin sheet design. The values of the behavior functions indicate that the optimum lies at the intersection of the gross buckling and local sheet buckling constraint surfaces.

Tables 1 and 2 are syntheses paths, i.e., complete histories of design evolution. Case 1-1 starts from the initial trial of

$$t_s = 0.3000 \qquad b_x = 5.000 \qquad t_w = 4.000$$

and after 15 design cycles converged to an upper relative minima of

$$t_s = 0.0919 \qquad b_x = 5.2393 \qquad t_w = 2.2545$$

It is interesting to note the correspondence of this intermediate point with the final output Point B of Case 1-1 as synthesized with the Monte Carlo program. The next alternate step

$$t_s = 0.2768 \qquad b_x = 5.2711 \qquad t_w = 0.5234$$

is in the thick sheet design region. The synthesis path continues in this same region of the design parameter space until it converges to the absolute minimum at

$$t_s = 0.2803 \qquad b_x = 4.3735 \qquad t_w = 0.0101$$

This final optimum is identical to the final output Point A of Case 1-1 as synthesized with the Monte Carlo program.

Cases 3-1, 3-2 and 3-3 are design problems where the waffle plates are subjected to three sets of load conditions. The final optimum designs are balanced designs for the three load condition system.

The motivation for developing a synthesis capability for multiple load conditions is twofold:

a. the optimum design may be controlled by constraints of more than one load condition (e.g. the optimum design may lie at the intersection of the gross buckling constraint of load condition No. 1 and local sheet buckling constraint of load condition No. 3.)

b. at the outset of most design problems involving a multiplicity of load conditions it is usually not possible to predict with certainty that a single load condition will dominate. In those cases where it is known that a single load condition will control the optimum design it is rarely possible to predict with confidence the dominant load condition

The task of establishing the existence of a dominant load condition, as well as the dominant load condition, is particularly complex since the design is not fixed. All of the programs generated for this study have the capability of handling five sets of design load systems. The programs consider the effects of all sets of loads and designs accordingly. The optimum designs of Cases 3-1, 3-2 and 3-3 are each controlled by a single load condition.

Compare, for each case, the results of starting from two trial design points. In general the corresponding coordinates of each set of optima agree within a few percent. The greatest discrepancy occurs in the b_x design parameter. It is important to note that b_x appears almost invariably in the denominator of the analysis expressions thus lending a hyperbolic character to its influence. In general, the design will be out in the flat portion of the hyperboloid. Therefore, there is not a strong dependency of the behavior and weight of this design parameter. The double points are considered sufficiently close to consider these results as the absolute minima and conclude that Compromise II successfully synthesizes waffle plates.

Table 2 is a synthesis path of Case 3-3. Starting from an initial trial of

$$t_s = 0.4000 \qquad b_x = 5.000 \qquad t_w = 4.000$$

the synthesis path converged to

$$t_s = 0.1018 \qquad b_x = 5.2467 \qquad t_w = 2.0453$$

after 15 design cycles. This point was found to be at the bottom of an upper relative minimum pocket. The next alternate step

$$t_s = 0.3383 \qquad b_x = 4.1035 \qquad t_w = 0.1746$$

move the design into the thick sheet design region. The synthesis path then converged to the thick sheet optimum

$$t_s = 0.3447 \qquad b_x = 4.0714 \qquad t_w = 0.0099$$

The program successfully distinguished between two relative minima where the difference in the weight was only 3 percent.

Waffle Design Concepts

The following is a list of comments generated as a result of studying the various optima. Whenever possible, comparison with preconceived notions is included.

Compare case 1-1, 1-2 and 1-3. Case 1-1 is constrained to a total depth, H of 0.4", and the optimum weight is 34.0346. If this requirement is relaxed to 0.6" as in Case 1-2, the optimum weight reduces to 16.8585 amounting to a weight savings of over 100 percent. If the total depth is further increased to 0.8, as in Case 1-3, the optimum decreases still further to 10.9423 amounting to an additional weight savings of over 50 percent. This conflict of weight versus total depth available for a stiffened panel often exists. The results presented here indicate dramatically the weight savings which can be accomplished by increasing the total depth of the structure.

Notice, too, that the basic nature of the design shifts with increasing H. When H is constrained to 0.4", the optimum design is a thick sheet design, but when H = 0.6" or 0.8" it is a thin sheet design of considerably lower weight.

When the total depth is severely limited (e.g., H = 0.4") it is found more advantageous to increase both the twisting and bending stiffness by adding to the sheet thickness rather than to increase just the bending stiffness by adding to the stiffeners.

Since the weight-strength character of waffle plates is so highly dependent upon H, it is suggested that in the future H be treated as a design parameter with appropriate upper and lower bounds. At first one might think that the optimum would always lie at the upper bound on H, but this is not necessarily so. As H increases, an additional design conflict becomes active. If H gets too large, the local stiffener buckling behavior becomes critical.

Case 1-3 is bounded by two active constraints, gross buckling and local sheet buckling. Local stiffener buckling will not be active because of the lower bound on t_w unless the total depth H is increased. It may be possible, although a case has not been found as yet, that an optimum design be bounded by all three buckling criterion.

Optimum shift and design type shift may occur as a consequence of a material change. The optimum design for Case 3-1 (aluminum) is a thick sheet design of weight 139.60 pounds. Moving up the density scale, Case 3-3 (titanium) has as its optimum a thick sheet design of weight 193.47. Notice from Table 2, cycle 15 that this case has a competitive thin sheet design of 196.97. Moving further up the density scale, Case 3-2 (steel) has a thin sheet optimum weight of 187.19. The Monte Carlo program was used, with a prejudiced starting point, to find the thick sheet relative minimum pocket for Case 3-2 (steel). The results were as follows:

$$t_s = 0.2788 \qquad b_x = 3.0260 \qquad t_w = 0.0118$$

$$W = 270.9432$$

These results clearly indicate that the optimum weight and design type may shift due to material change.

CONCLUSIONS

The conclusions may be stated simply as follows:

Based on the analysis presented in Appendix B the successful development of a synthesis capability for symmetric waffle plates with integral orthogonal stiffeners is reported.

It is thought that the method developed may be applied with minor modifications to a wide variety of systems with a non-linear merit function, regardless of the existence of relative minima. Completion of the symmetric waffle synthesis program supports the contention that a structural synthesis capability can be developed for complex structural systems of current and future importance.

A secondary set of conclusions are derived and appear in the form of recommendations for future work on analysis and synthesis.

The background study in preparation for developing a synthesis process is essentially a study of the existing technology and therefore points up the shortcomings and absence of applicable analyses. The following is a list of aspects of the analysis which merit further study:

a. Study of the interaction expression for three inplane loads over a wide range of aspect ratios.

b. A study to more clearly define the range of applicability of the equivalent plate concept. Further study to determine the appropriate buckling pattern when the equivalent plate theory is not applicable.

c. A study of the assumed boundary conditions used for the local buckling criterion of both stiffener and sheet.

d. A study of the stress distribution in the stable waffle plate based on a theory of elasticity solution.

Based on the results of the synthesis study, the following is a set of recommendations for future work on synthesis:

a. Use the current version of Compromise II to examine further the characteristics of the optimum design.

b. Develop more conclusive techniques for proving an optimum.

c. Develop and test more efficient methods of travel.

d. Increase the number of design parameter to six by permitting an unsymmetric stiffener configuration.

Engineering Division
Case Institute of Technology
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APPENDIX A

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APPENDIX B

GOVERNING TECHNOLOGY OF SYMMETRIC WAFFLE PLATES

The first step in developing a procedure for minimum weight balanced design of integrally stiffened waffle plates is to put together a method of analysis which adequately predicts the behavior of such plates. The governing technology to be used to develop a synthesis capability for integrally stiffened waffle-like plates is given in this Appendix. While no detailed derivations are presented, the specialization of expressions to apply to the synthesis of orthogonally stiffened waffle plates is outlined. A discussion of the assumptions and restrictions imposed by the analysis is incorporated into the presentation of the analysis.

Orthotropic Plate Equations

Many researchers, (see Bibliography, Appendix A), have studied individual characteristics of integrally stiffened flat plates, cylinders and curved panels. Dow, Libove, and Hubka[2] derive the formulas for the elastic constants of the equivalent orthotropic plate. This analysis facilitates the use of orthotropic plate theory for the class of flat structures known as integrally stiffened waffle-like plates. Since the investigation is restricted to flat rectangular waffle plates with simply supported edges, subject to any combination of inplane loads N_x, N_y and N_{xy}, the governing differential equation is:

$$D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} = 0 \tag{B1}$$

The gross instability of the equivalent plate is studied employing a linear elastic buckling technology and the assumed mode technique and hence can be viewed as an eigenvalue problem.

Gross Buckling of a Waffle Plate

The gross buckling behavior of the waffle plate is studied by first transforming the waffle plate to its equivalent orthotropic plate via the elastic constants and then employing a linear elastic analysis to determine the buckling loads. The assumed mode used throughout the study is:

$$w(x) = \sum_n \sum_m A_{mn} \sin \frac{n \pi x}{a} \sin \frac{m \pi y}{b} \qquad (B2)$$

which was reduced from the general trigonometric Fourier Series.

The interaction expressions for several cases of isotropic plates subject to combinations of two inplane loads can be found in the literature. Bleich(3) and Gerard(4) give expressions for flat isotropic plates with moderate aspect ratios $\alpha_r = a/b$.

These interaction equations enjoy substantial experimental support over the range of aspect ratio

$$\frac{1}{3} < \alpha_r < 3$$

Lekhnitski(5) gives the same interaction expression for orthotropic plates subject to combinations of two inplane loads.

A single interaction expression can be conjectured from these interaction expressions.

$$\frac{N_x}{(N_x)_{cr}} + \frac{N_y}{(N_y)_{cr}} + \left[\frac{N_{xy}}{(N_{xy})_{cr}} \right]^2 = 1 \qquad (B3)$$

It is readily apparent that this expression reduces to the accepted interaction expression for any combination of two inplane loads, applied to flat orthotropic and isotropic plates.

Since the combined interaction expression reduces to all possible sub-cases it may be considered as a proposed interaction formula for flat orthotropic plates subject to any combination of three inplane loads N_x, N_y and N_{xy}. A first analysis, using the differential equations (B1) and assumed modes (B2), has been completed. Although only a few terms were retained in the series, the results gave support to equation (B3) as the correct expression. While the proposed interaction expression (B3) shows promise it is not firmly established as yet.

Since any synthesis process must be based on accepted analysis, the composite interaction expression (B3) can currently be used with confidence only for design load systems made up of any combination of two loads. It is intended that the synthesis capability developed be based on (B3) and apply for design load systems made up of any combination of two loads. At the same time this synthesis capability, based on the most plausible analysis available, will be capable of handling design load systems made up of three inplane forces.

In order to use the interaction expression (B3) it is necessary to have expressions for the individual critical loads $(N_x)_{cr}$, $(N_y)_{cr}$, and $(N_{xy})_{cr}$.

Instability of Orthotropic Plates Subject to a Single Inplane Load N_x, N_y or N_{xy}.

Expressions for the buckling loads of orthotropic plates subject to the single inplane loads, N_x, N_y and N_{xy}, are needed for the interaction expression (B3). The general method of attack, outlined above, is employed throughout using the differential equation (B1) with the proper applied load terms deleted and the assumed mode (B2).

Lekhnitski[5] gives the expression for orthotropic plate buckling under normal load N_x. Notice that the expression for $(N_y)_{cr}$ can also be derived from the expression for $(N_x)_{cr}$ by a simple permutation of subscripts and an exchange of $\alpha_r' = 1/\alpha_r$. The expression for N_x is as follows:

$$(N_x)_{cr} = - \frac{\pi^2 \sqrt{D_1 D_2}}{b^2} \left[\sqrt{\frac{D_1}{D_2}} \left(\frac{m'}{\alpha_r}\right)^2 + \frac{2 D_3}{\sqrt{D_1 D_2}} + \sqrt{\frac{D_2}{D_1}} \left(\frac{\alpha_r}{m'}\right)^2 \right] \tag{B4}$$

where m' is the integer yielding to the smallest value of N_x.

A simple permutation of subscripts and interchange of variables yields the following:

$$(N_y)_{cr} = - \frac{\pi^2 \sqrt{D_1 D_2}}{a^2} \left[\sqrt{\frac{D_2}{D_1}} (p' \alpha_r)^2 + \frac{2 D_3}{\sqrt{D_1 D_2}} + \sqrt{\frac{D_1}{D_2}} \left(\frac{1}{p' \alpha_r}\right)^2 \right] \tag{B5}$$

where p' is the integer yielding the smallest value of N_y.

Notice that both of the expressions will yield negative values for $(N_x)_{cr}$ and $(N_y)_{cr}$. This is a consequence of the sign convention assumed for the applied loads (See Fig 3).

Seydel[6] presents a study of the instability of simply supported orthotropic plates under an inplane shear loading. The following notation is used as a means of simplifying the expressions for shear buckling

$$\theta = \sqrt{\frac{D_1 D_2}{D_3^2}}$$

The following definitions apply only for $\theta > 1$ which is always true for waffle plates.

$$\beta = \frac{b}{a} \sqrt[4]{\frac{D_1}{D_2}} \qquad \varphi(m, n) = (m\beta)^4 + 2\,\frac{(m\beta)^2\, n^2}{\theta} + n^4$$

$$N_{xy} = + C_a \frac{\sqrt[4]{D_1 D_2{}^3}}{\left(\frac{b}{2}\right)^2} \tag{B6}$$

where m and n are integer parameters of the assumed mode expansion.

One additional restriction is that the value of β must always be between zero and unity. Since D_1 equals D_2 this means that the problem must always be cast such that the aspect ratio, a/b, be greater than unity.

C_a is a buckling coefficient given by the following expressions: for $n = 1, 2, 3$ and $m = q, q+1, q+2$.

Case I

Symmetric buckling with q odd

or

Antisymmetric buckling with q even

$$C_a = \frac{\pi^4}{128} \left\{ \frac{\varphi(q+1, 2)}{2(q+1)\beta} \right\}^{1/2} \left\{ \left[\frac{q}{2q+1}\right]^2 \left[\frac{1}{9\varphi(q, 1)} + \frac{9}{25\varphi(q, 3)} \right] + \left[\frac{q+2}{2q+3}\right]^2 \left[\frac{1}{9\varphi(q+2, 1)} + \frac{9}{25\varphi(q+2, 3)} \right] \right\}^{-1/2} \tag{B7}$$

Case II

Symmetric buckling with q even

or

Antisymmetric buckling with q odd

$$C_a = \frac{\pi^4}{128} \left\{ \frac{1}{2(q+1)\beta} \right\} \left\{ \left[\frac{1}{9\varphi(q+1, 1)} + \frac{9}{25\varphi(q+1, 3)} \right] \right.$$

$$\times \left[\frac{q^2}{(2q + 1)^2 \varphi(q, 2)} + \frac{(q + 2)^2}{(2q + 3)^2 \varphi(q + 2, 2)} \right] \Bigg\}^{-1/2} \qquad (B8)$$

Notice that there exists a unique expression for C_a for both symmetric and antisymmetric buckling. Stein and Neff[7a] showed that either the symmetric or antisymmetric mode can be critical depending upon α_r, the aspect ratio. This is also true for orthotropic plates but the dependency is upon β where $\beta = \alpha_r' \sqrt[4]{D_1/D_2}$. The critical value of N_{xy} must therefore be determined from two complete sets of mode parameters. That is, given the values of θ and β as data, determine the minimum magnitude of N_{xy} as a function of the integer parameter q and gross buckling pattern (symmetric or antisymmetric).

The above expressions are used in conjunction with the interaction expression (B3) to give the necessary technology to predict the gross buckling behavior of simply supported waffle plates. This analysis constitutes a major portion of the governing technology upon which the synthesis capability is based. Up to this point the analysis has dealt only with orthotropic plates, in general. In order to use the analysis it is necessary to transform the waffle plate into an equivalent orthotropic plate. The equations of the elastic constants used for this transformation were derived by Dow, Libove and Hubka[2].

Elastic Constants for the Equivalent Plate

The gross buckling phenomenon of the waffle plate is studied by generating an equivalent plate and studying the buckling characteristic of this equivalent plate. A detailed derivation of the transformation is outlined in Ref. 2. The basic assumptions or restrictions necessary to make this transformation are as follows:

1. The rib spacings of the integrally-stiffened plate are small in comparison with the overall width and length of the plate. This assumption is made in order that the average or overall behavior may be studied rather than a detailed study of any particular segment. Several studies have been made where the rib spacing was large (i.e., one, two or even three ribs in either or both directions). In this case the behavior cannot be examined on a gross scale but a detailed elastic stability study of the interacting subsystems has to be performed.

2. This particular analysis is concerned with waffle plates with only longitudinal and transverse ribs. The expressions for the elastic constants are specialized from those given in Ref. 1. It is important to notice that the restriction in no way limits the gross instability analysis. In order to extend this study to include skewed stiffeners, only a modification of the expressions for the elastic constants would be necessary.

3. Since the shear stress, τ_{xy}, is zero at the outer boundary of each stiffener, it is assumed that the shear stress τ_{xy} is zero throughout each stiffener. The total shear load therefore must be carried by the back-up sheet.

4. The formulas for the equivalent elastic constants involve coefficients α, β, α' and β' which define the effectiveness of a rib in resisting transverse stretching and bending and inplane shearing and twisting.

The terms β and β' represents that fraction of the volume of a rib resisting stretching and shearing respectively. The terms α and α' locate the centers of gravity of these effective volumes. The lower bound of zero is assumed for β and β' and consequently the values of α and α' are immaterial.

Consider a waffle plate with only one set of stiffeners, subject to a load N_x which is transverse to the stiffeners. It is readily apparent that the normal stress in the sheet midway between two ribs is N_x/t if the stiffener spacing is large compared to the stiffener thickness. It is also true that the normal stress in the vicinity of a rib is lower than the normal stress midway between two ribs. However, a uniform normal stress equal to that at the midpoint is assumed to exist throughout. That is, it is assumed that a rib has no effect on the stress distribution generated by a transverse load.

With the above restrictions, consider the elastic constants used in the previous section to study the gross buckling phenomenon. The expressions for the flexural rigidities and torsional rigidity for the symmetric waffle are:

$$D_1 = D_2 = EH^3 \left\{ \frac{1}{12(1-\mu^2)} \left(\frac{t_s}{H}\right)^3 + \frac{1}{12}\left(1 - \frac{t_s}{H}\right)^3 \left(\frac{t_w}{b_x}\right) \right.$$

$$+ \frac{\mu}{8(1-\mu^2)} \left(1 - \frac{t_s}{H}\right)\left(\frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right) \left[\frac{(1+\mu)}{\frac{\mu}{1-\mu}\left(\frac{t_s}{H}\right) + \mu\left(1 - \frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)} \right.$$

$$\left. \left. + \frac{(1-\mu)}{\frac{\mu}{1+\mu}\left(\frac{t_s}{H}\right) + \mu\left(1 - \frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)} \right] \right\} \qquad \text{(B9)}$$

and

$$2\,D_3 = \frac{E\,H^3}{2}\left\{\frac{1}{3\,(1-\mu^2)}\left(\frac{t_s}{H}\right)^3 + \frac{\left[\frac{\mu}{(1-\mu^2)}\left(\frac{t_s}{H}\right)\left[\left(1-\frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)\right]\right]^2}{\left[\frac{1}{(1-\mu^2)}\left(\frac{t_s}{H}\right)+\left(1-\frac{t_s}{H}\right)\left(\frac{t_w}{b_x}\right)\right]^2-\left[\frac{\mu}{(1-\mu^2)}\left(\frac{t_s}{H}\right)\right]^2}\right\} \tag{B10}$$

<u>Local Buckling of a Waffle Plate</u>

Two distinct modes of local buckling of waffle plates are considered. Examine the case where the stiffeners are thin compared to the sheet. Then it is possible for the sheet to undergo a stable inplane displacement while the stiffeners become unstable in a mode comparable to flange buckling. On the other hand, if the sheet is thin compared to the stiffeners, it is possible for each individual panel to buckle as a rectangular plate while the stiffeners undergo a stable inplane deflection. In both cases the mathematical model is a rectangular isotropic plate.

Consider the case of stiffener instability which occurs when the stiffener thickness is small compared to the sheet thickness.

The model used for analysis is the portion of a longitudinal stiffener between two adjacent transverse stiffeners. The boundary conditions are taken as follows: hinged ends, hinged on one edge and free on the other edge. The applied load is a uniformly distributed normal load applied along the hinged ends of the plate.

The expression for critical loads are as follows:

$$(\bar{N}_x)_{cr} = -\pi^2 D' \left(\frac{t_w}{H-t_s}\right)^2 \left[\frac{b_x\,t_s + t_w\,(H-t_s)}{b_x}\right] K_B \tag{B11}$$

and

$$(\bar{N}_y)_{cr} = -\pi^2 D' \left(\frac{t_w}{H-t_s}\right)^2 \left[\frac{b\,t_s + t_w\,(H-t_s)}{b_x}\right] K_B \tag{B12}$$

where:

$$K_B = \left(\frac{1}{\varphi}\right)^2 + 0.425 \tag{B13}$$

$$\varphi = \frac{b_x - t_w}{H - t_s}$$

$$D' = \frac{E}{12\,(1 - \mu^2)}$$

Consider the case of sheet instability which occurs when the sheet thickness is small compared to the stiffener thickness. The mathematical model used for this system is a rectangular isotropic plate bounded by two transverse stiffeners and two longitudinal stiffeners which are assumed to provide hinged boundary conditions. Since it is possible for the sheet to carry all three inplane loads N_x, N_y and N_{xy}, the interaction expression

$$\frac{N_x}{(\tilde{N}_x)_{cr}} + \frac{N_y}{(\tilde{N}_y)_{cr}} + \left[\frac{N_{xy}}{(\tilde{N}_{xy})_{cr}}\right]^2 = 1 \qquad \text{(B14)}$$

must be used to predict the instability of the back-up sheet.

For this case the expressions for $(\tilde{N}_x)_{cr}$, $(\tilde{N}_y)_{cr}$ and $(\tilde{N}_{xy})_{cr}$ are as follows:

$$(\tilde{N}_x)_{cr} = -4.00\,\frac{\pi^2 D}{t_s\,(b_x - t_w)^2}\left[\frac{t_s\,b_x + t_w\,(H - t_s)}{b_x}\right] \qquad \text{(B15)}$$

$$(\tilde{N}_y)_{cr} = -4.00\,\frac{\pi^2 D}{t_s\,(b_x - t_w)^2}\left[\frac{t_s\,b_x + t_w\,(H - t_s)}{b_x}\right] \qquad \text{(B16)}$$

$$(\tilde{N}_{xy})_{cr} = \pm 9.34\left[\frac{\pi^2 E\,t_s^3}{12\,(1 - \mu^2)\,b_x - t_w)^2}\right] \qquad \text{(B17)}$$

The above expressions give the equations necessary to predict local waffle plate instability when it occurs as stiffener or sheet buckling. The one remaining mode of failure is the material yield criterion which is presented in the next section.

Material Yield Criterion

Under normal applications the design of waffle plates is governed by a buckling criterion. Since the synthesis process is developed from a linear theory of elastic buckling, it is necessary to know when the final design is such that the analysis is not valid and an inelastic buckling theory should be employed. A material yield criterion is used as an alarm to determine when the final design is not governed by the elastic buckling constraints. Because of the nature of the synthesis process, it is possible for an intermediate design to be governed by material yield criterion. If the re-design process continues and finds a new design which is governed by an elastic buckling constraint, the intermediate design is subsequently discarded. In short, the analysis assumes ideal elastic-plastic behavior of the structural material.

The distortion energy criterion, employed as the material yield alarm, is as follows:

$$\left\{ \sigma_x^2 - \sigma_x \sigma_y + \sigma_y^2 + 3\tau_{xy}^2 \right\}^{\frac{1}{2}} = Y \qquad (B18)$$

The expressions for stresses are substituted into (B18) to give

$$Y = \frac{1}{H} \left\{ \frac{< N_x^2 - N_x N_y + N_y^2 >}{\left[(\frac{t_s}{H}) + (1 - \frac{t_s}{H}) \frac{t_w}{b_x} \right]^2} + 3 \left(\frac{N_{xy}}{t_s/H} \right)^2 \right\}^{\frac{1}{2}} \qquad (B19)$$

The ordering of principal stresses is unnecessary and the above expression is valid regardless of the magnitude or sign of the applied loads.

A uniaxial state of stress exists in each stiffener and the distortion energy criterion reduces to:

$$Y = \sigma_{max} - \sigma_{min}$$

Since the coefficient relating stress to load is always positive the stiffener yield criteria can be expressed as

$$Y = |N_x| \left[\frac{b_x}{t_s b_x + t_w d} \right] \qquad (B20)$$

$$Y = |N_y| \left[\frac{b_x}{t_s b_x + t_w d} \right] \qquad (B21)$$

for the symmetric configuration of stiffeners.

The foregoing expressions (B19) (B20) and (B21) give the analysis necessary to predict material yield.

Side Constraints

Any assumption or restriction which is not imposed automatically by the analysis must be provided for as an independent side constraint in the synthesis process, as is done in any design process. For example, in order to use the equivalent elastic constants, it was assumed that the rib spacing is small in comparison with the overall length and width of the plate. The analysis, as such, does not impose this restriction. Therefore, it must be incorporated into the synthesis process as

$$b_x < \frac{a}{K_1} \qquad \text{(B22)}$$

$$b_x < \frac{b}{K_1} \qquad \text{(B23)}$$

where K_1 is a constant greater than unity and directly related to the minimum number of stiffeners.

In preparing a synthesis process, all mathematically possible but physically absurd designs must be anticipated and prevented through the use of independent side constraints. For example, it is physically impossible for the stiffener spacing to be less than the stiffener thickness, but there is no mathematical restriction inherent to the analysis imposing this requirement. Therefore, a lower bound is placed on the stiffener spacing as:

$$b_x > t_w$$

A second lower bound is placed on b_x. Due to production considerations it may be impossible to produce a waffle plate with a stiffener spacing less than some fixed value regardless of the dimension of t_w. Therefore, a fixed lower bound of K_5 is also used and the mathematical statement of this lower bound is as follows:

$$b_x > K_5$$

$$b_x > t_w$$

Only one of these statements is necessary for if $K_5 > t_w$ and $b_x > K_5$ then it follows that $b_x > t_w$. The procedure is to select the larger of K_5 and t_w and use this as a lower bound, b_L.

$$b_x > b_L \qquad \text{(B24)}$$

Another example of a compatability bound arises because of the absence of an inherent upper bound on the sheet thickness. Mathematically it is possible for the sheet thickness to become greater than the total depth of stiffener plus sheet, i.e., the stiffeners assume a negative depth. Therefore, an upper bound is incorporated as

$$t_s < H \tag{B25}$$

to prevent a design which is physically absurd.

Independent side constraints can arise from restrictions external to the waffle analysis. For example, if the smooth side of the waffle is to be used as an aerodynamic surface, it would be undesirable for the sheet thickness to vanish. Therefore, a lower bound is placed on the sheet thickness as

$$t_s > K_2 \tag{B26}$$

where K_2 is an arbitrary constant.

If K_2 is set to zero, this lower bound can be considered as a constraint which prevents a physically absurd design i.e., negative sheet thickness.

A lower bound is also placed on the stiffener thickness as

$$t_w > K_3 \tag{B27}$$

where K_3 is an arbitrary constant.

If K_3 is set to zero it prevents a physically absurd design, i.e. negative stiffener thickness.

The need for some of the above mentioned side constraints is inherent to the synthesis concept. The mathematical formulation of an analysis to be used solely to predict the behavior of a structure may properly assume the existence of a physically attainable structure. However, when an analysis is to be used as a component part of a synthesis process, wherein redesign is to take place automatically, care must be exercised in assuring that only physically attainable designs are permitted. This then is one role played by what have been called side constraints.

Development of a synthesis capability for orthogonally stiffened waffle plates based on the technology presented herein will deal with design parameters which are actual dimensions of the structure, consider the effects of local and over-all instability on the behavior, and include a substantial number of side constraints. A flow chart summarizing the analysis used as a component part of the synthesis program is shown in Fig. 10.

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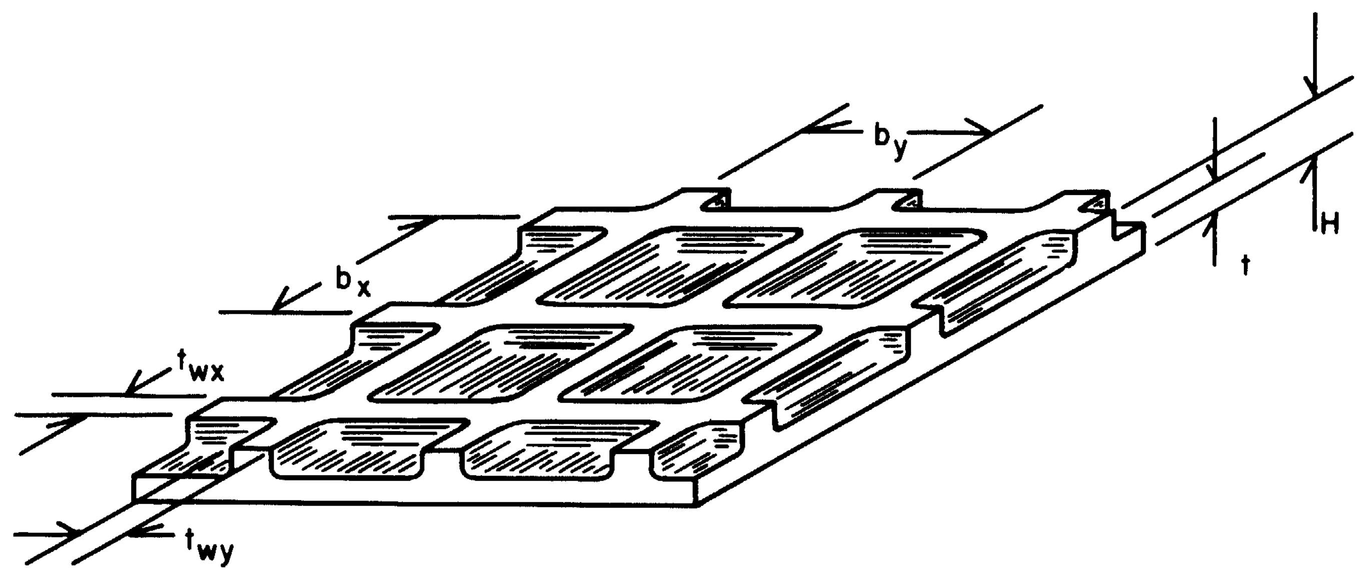
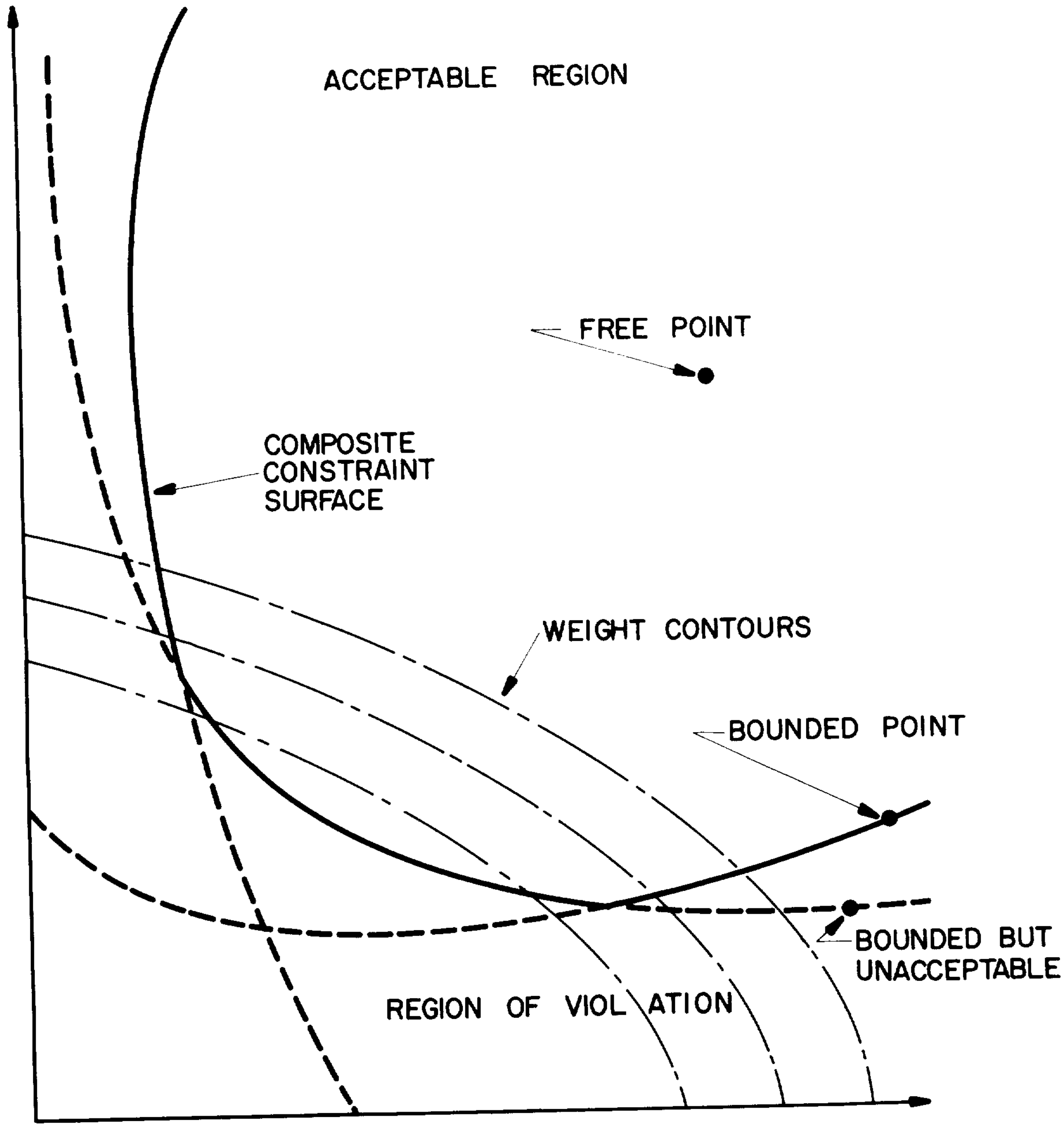


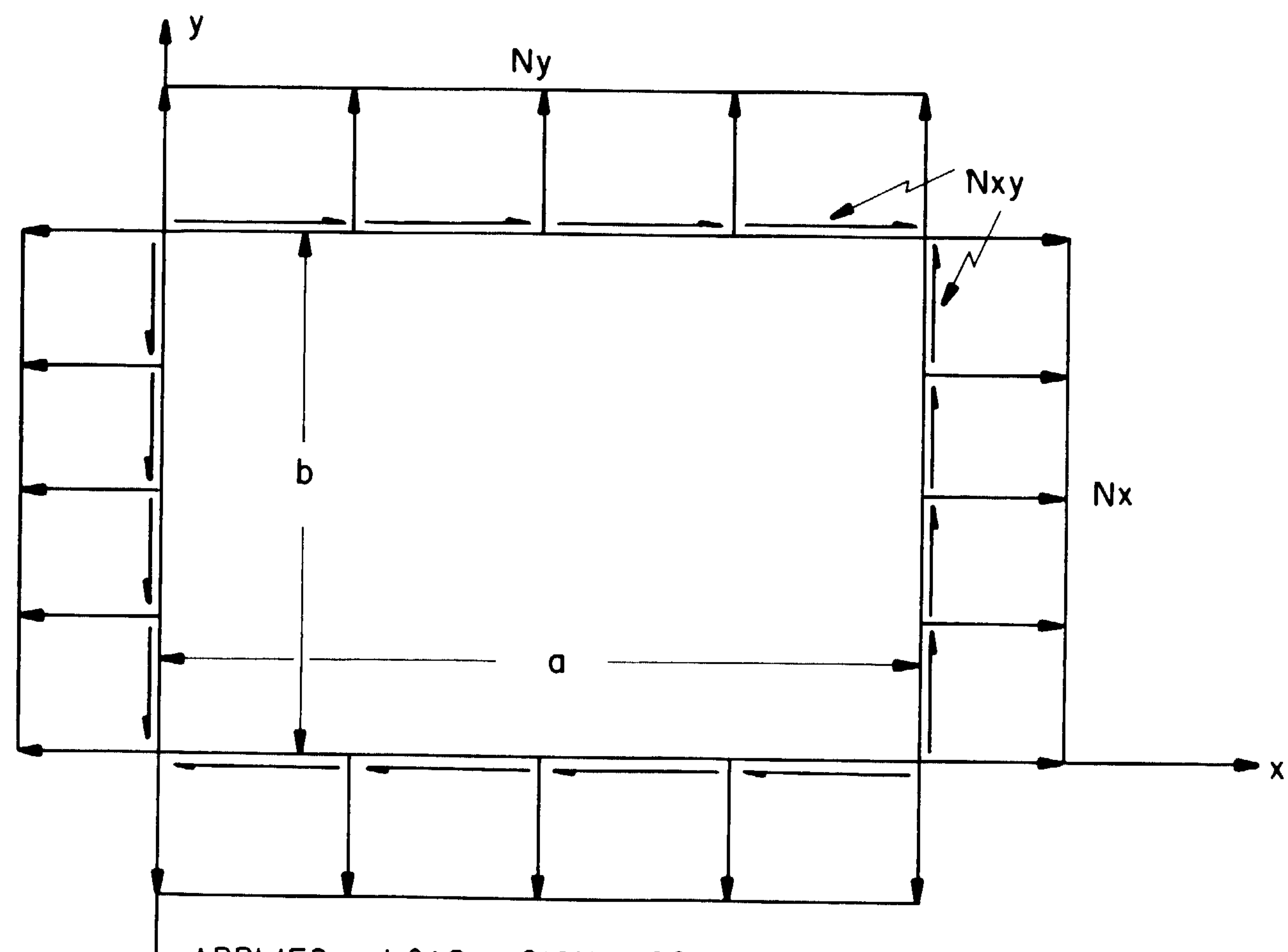
INTEGRALLY STIFFENED WAFFLE - LIKE PLATE

FIGURE 1



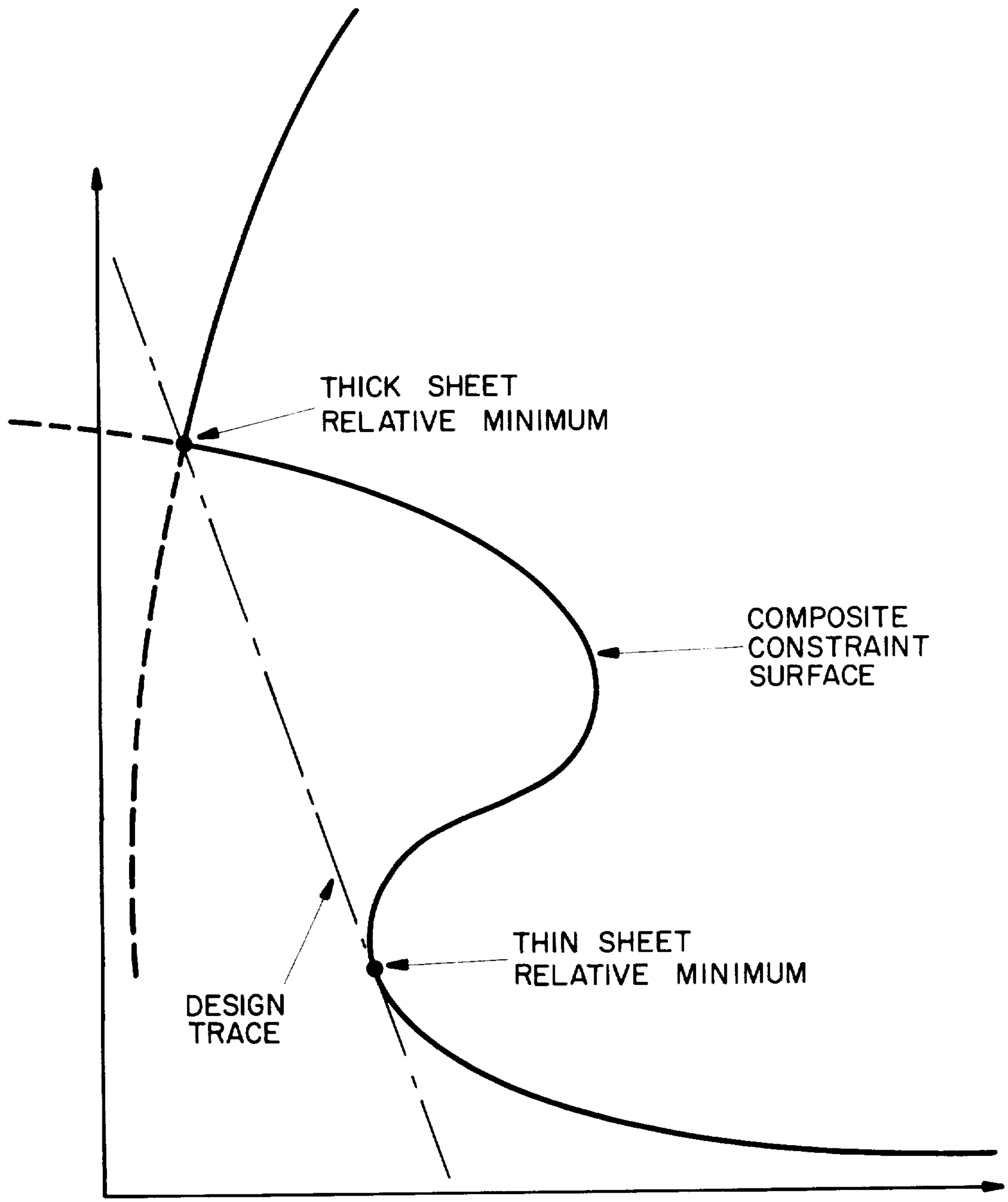
DESIGN PARAMETER SPACE NOMENCLATURE

FIGURE 2



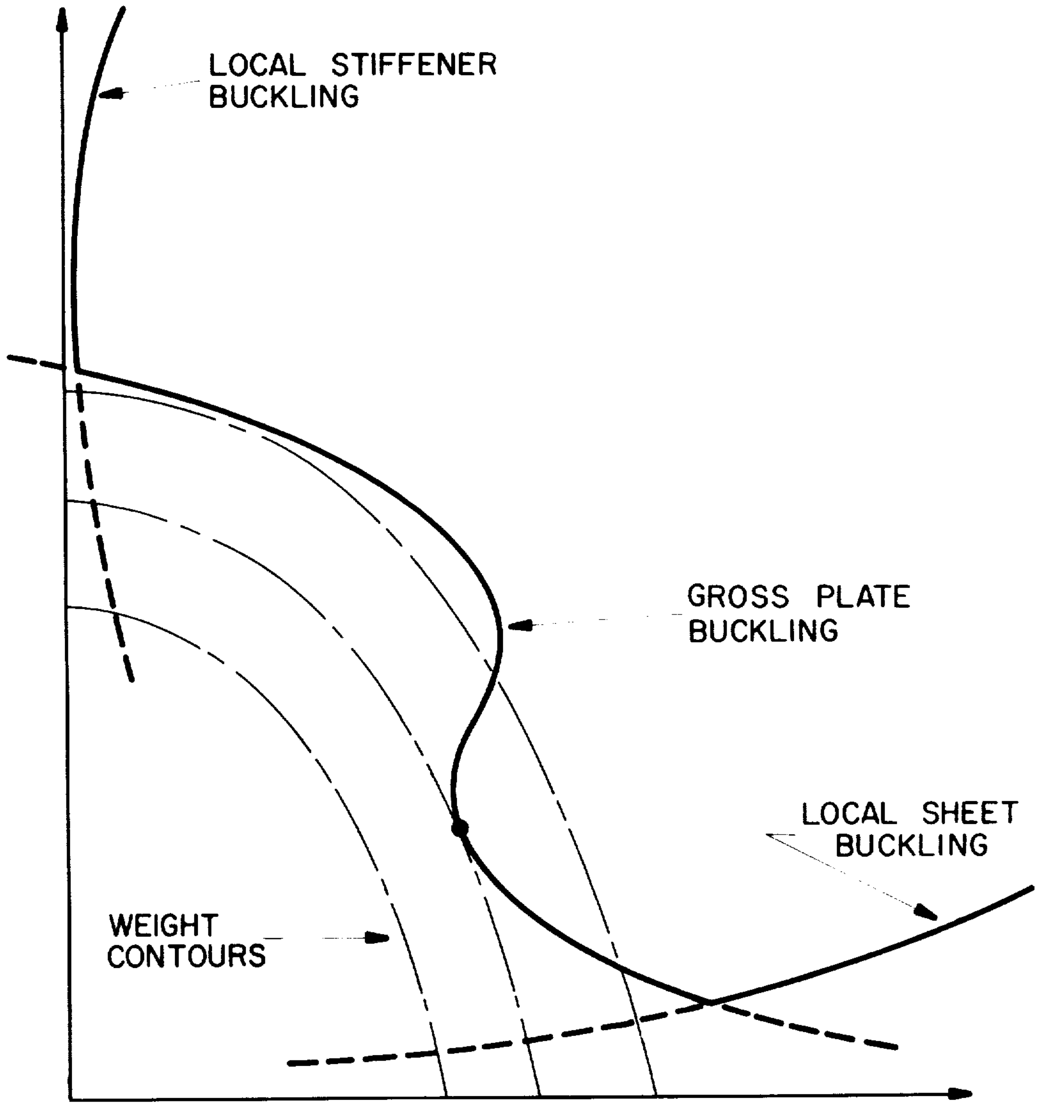
APPLIED LOAD SIGN CONVENTION

FIGURE 3



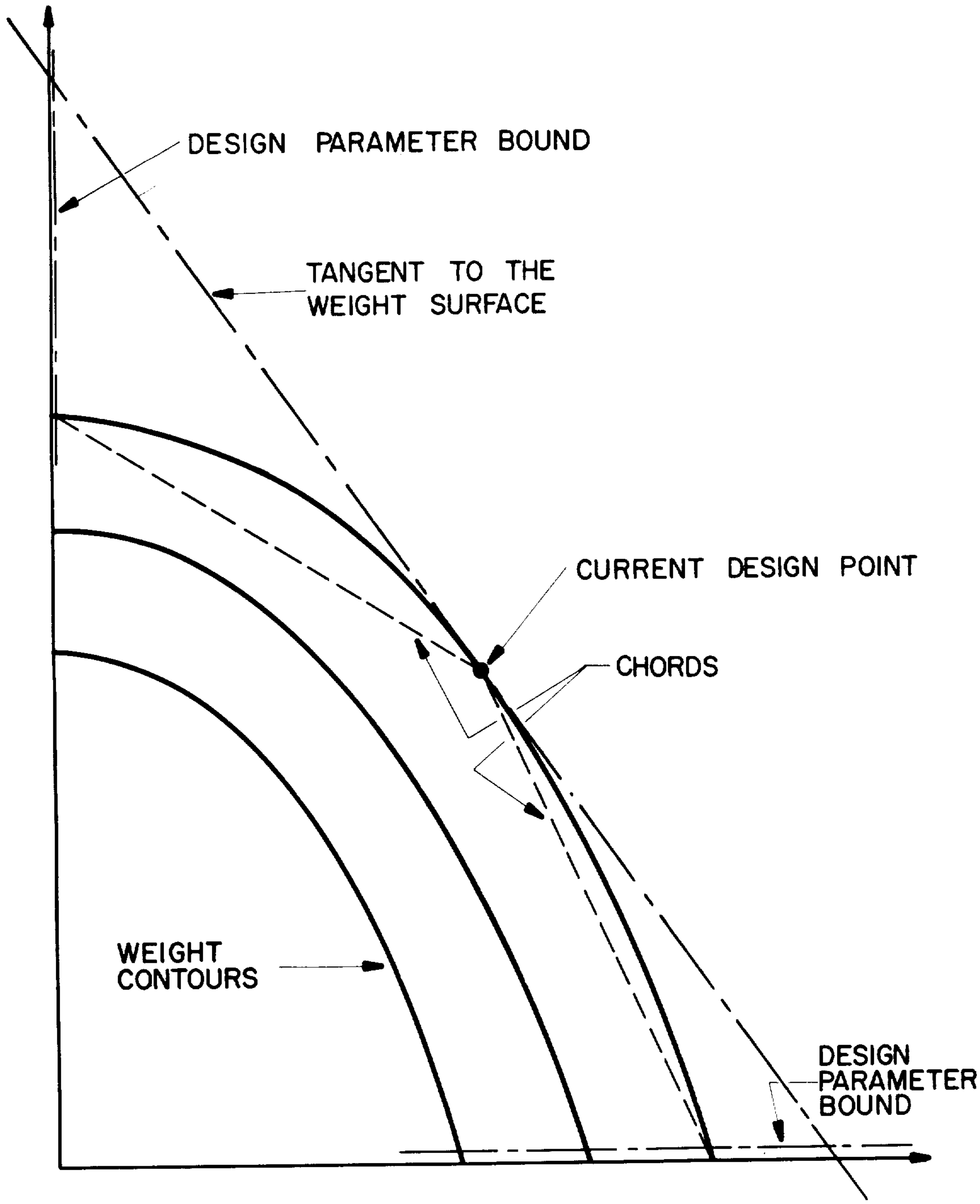
DESIGN TRACE FROM ONE RELATIVE MINIMA TO ANOTHER SHOWING PIERCING OF THE COMPOSITE CONSTRAINT SURFACE

FIGURE 4



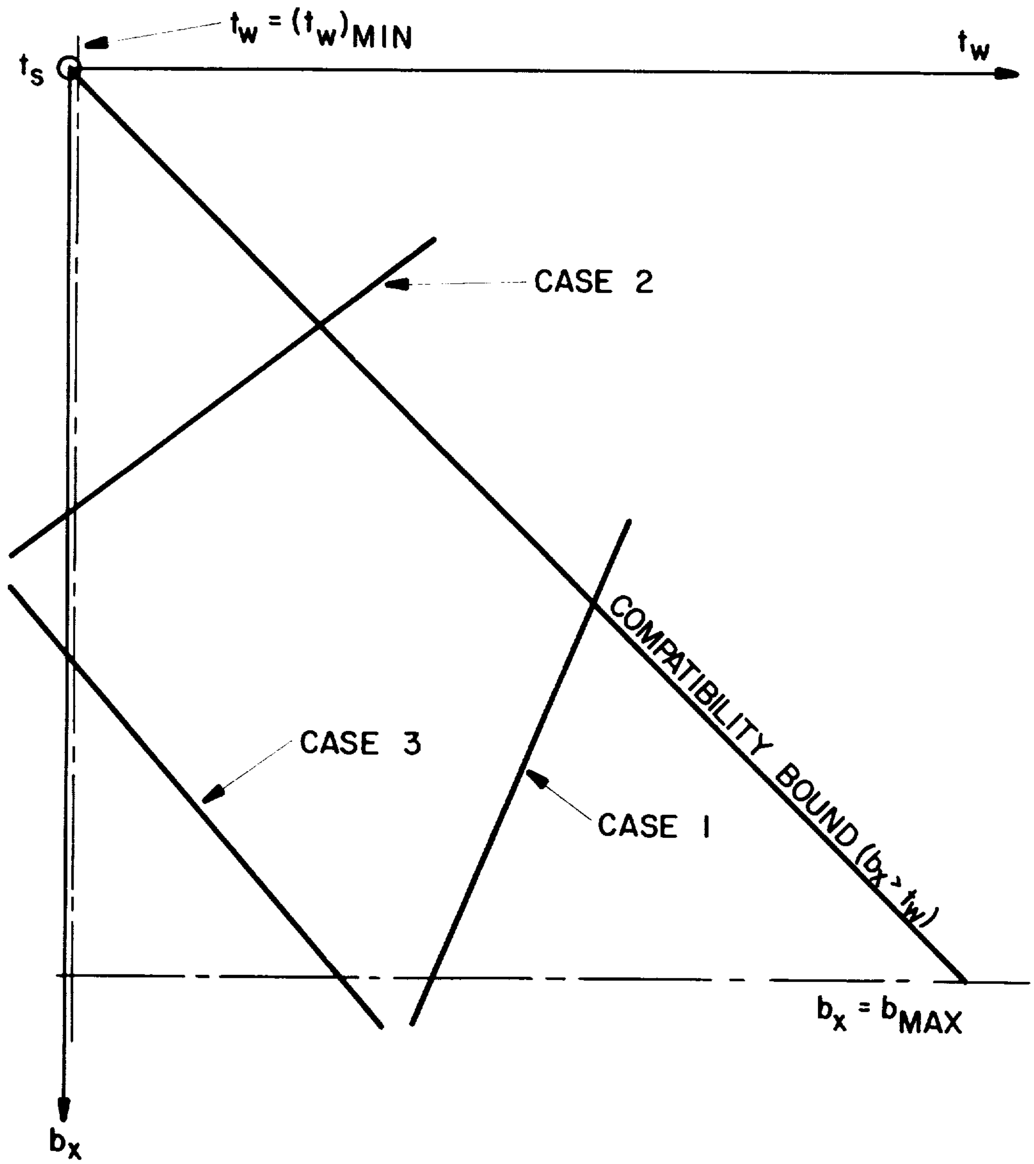
RANDOM SLICE THROUGH A DESIGN PARAMETER SPACE

FIGURE 5



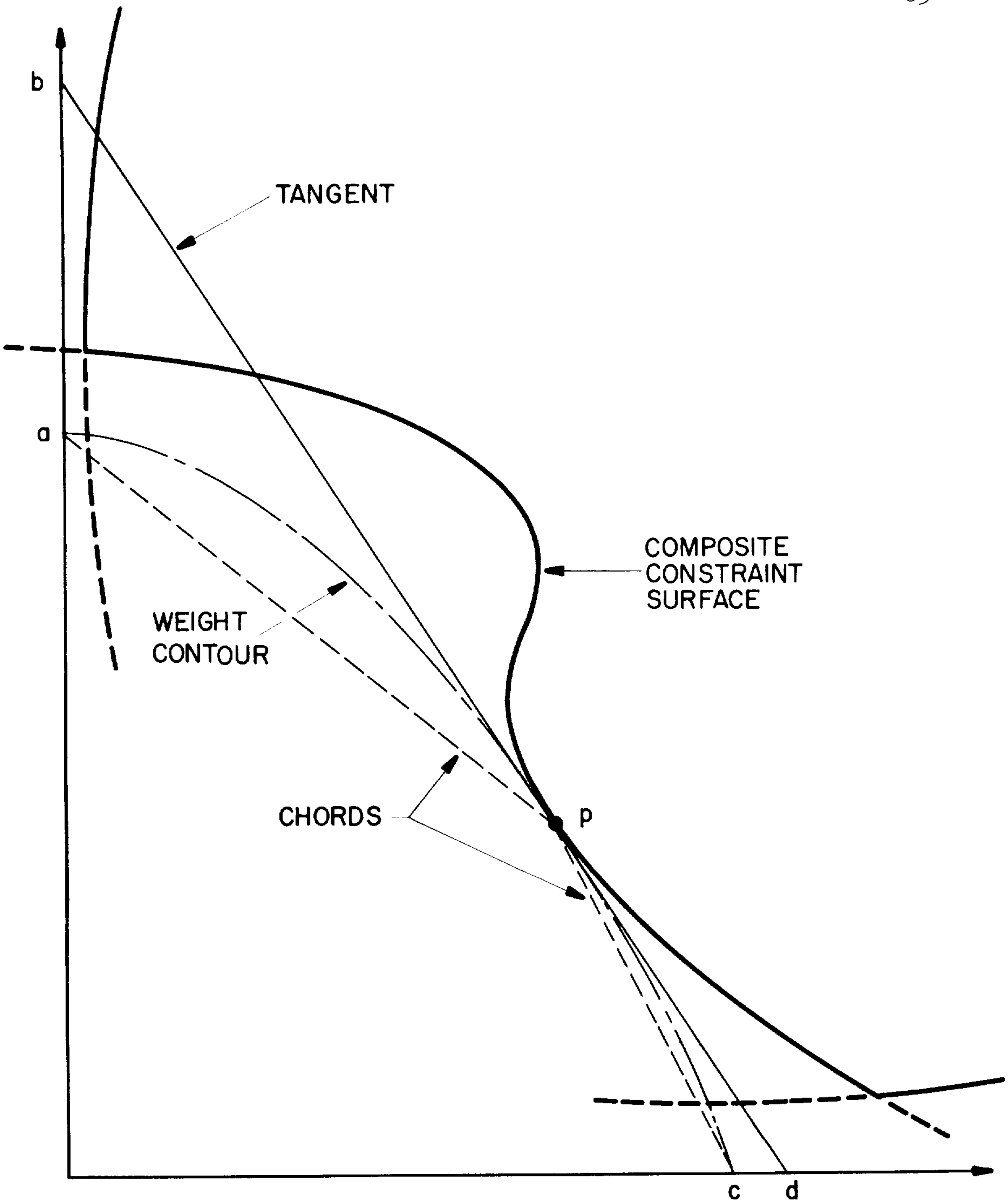
ACCEPTABLE ALTERNATE STEP REGIONS IN THE RANDOM PLANE

FIGURE 6

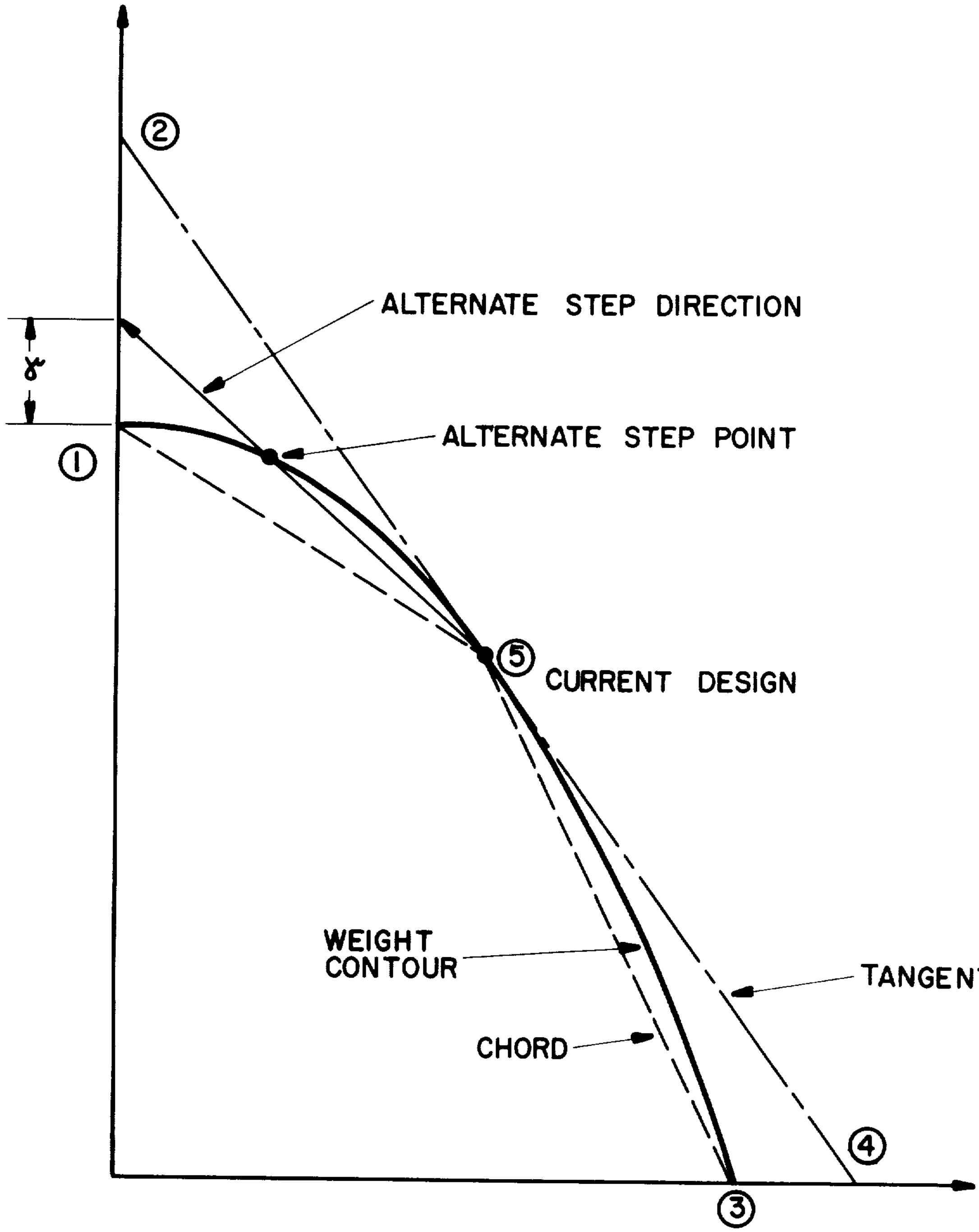


TYPES OF RANDOM PLANES

FIGURE 7



HYPOTHETICAL DESIGN PARAMETER SPACE FIGURE 8



COMPROMISE II - ALTERNATE STEP

FIGURE 9

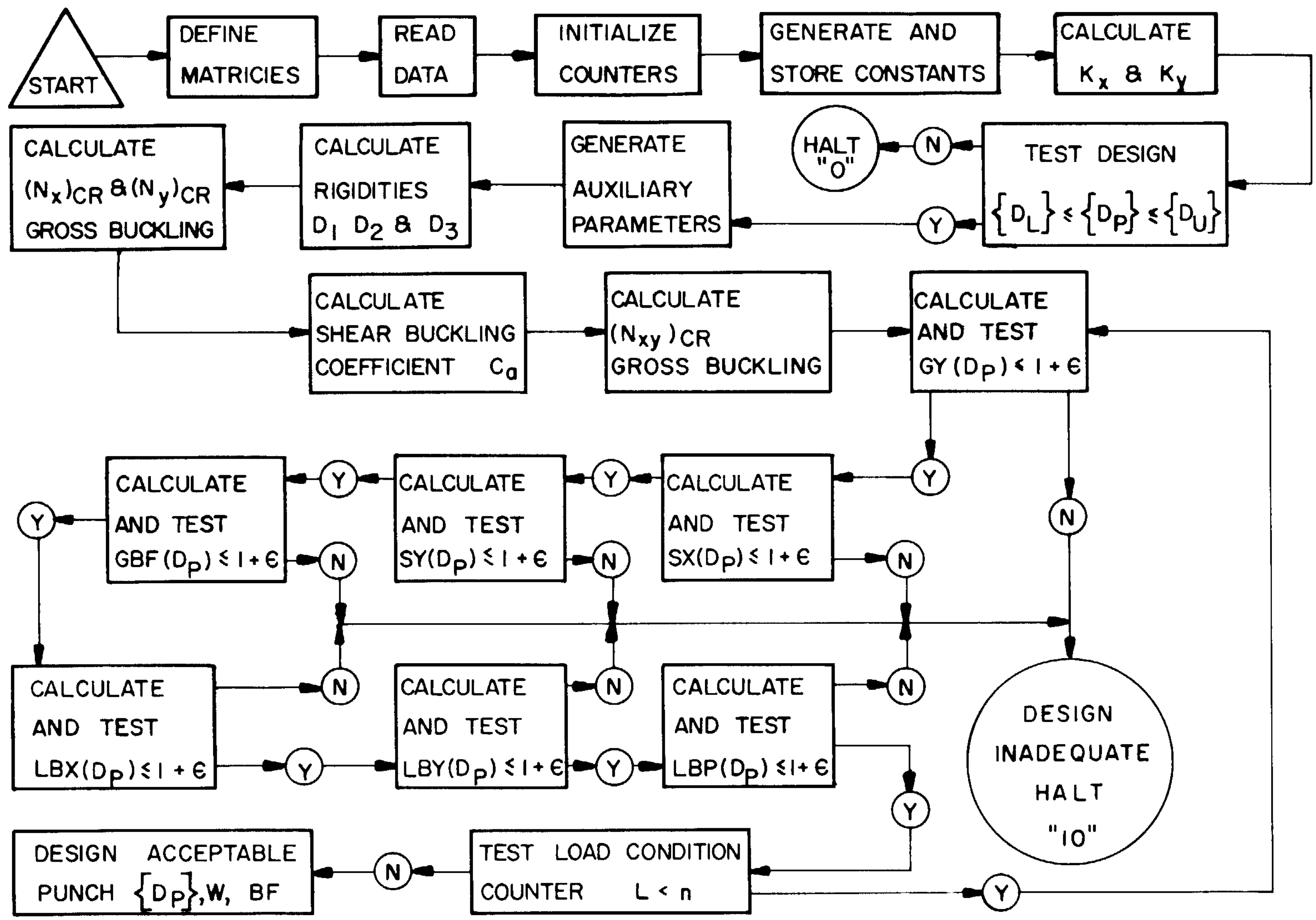


FIGURE 10 WAFFLE ANALYSIS FLOW DIAGRAM

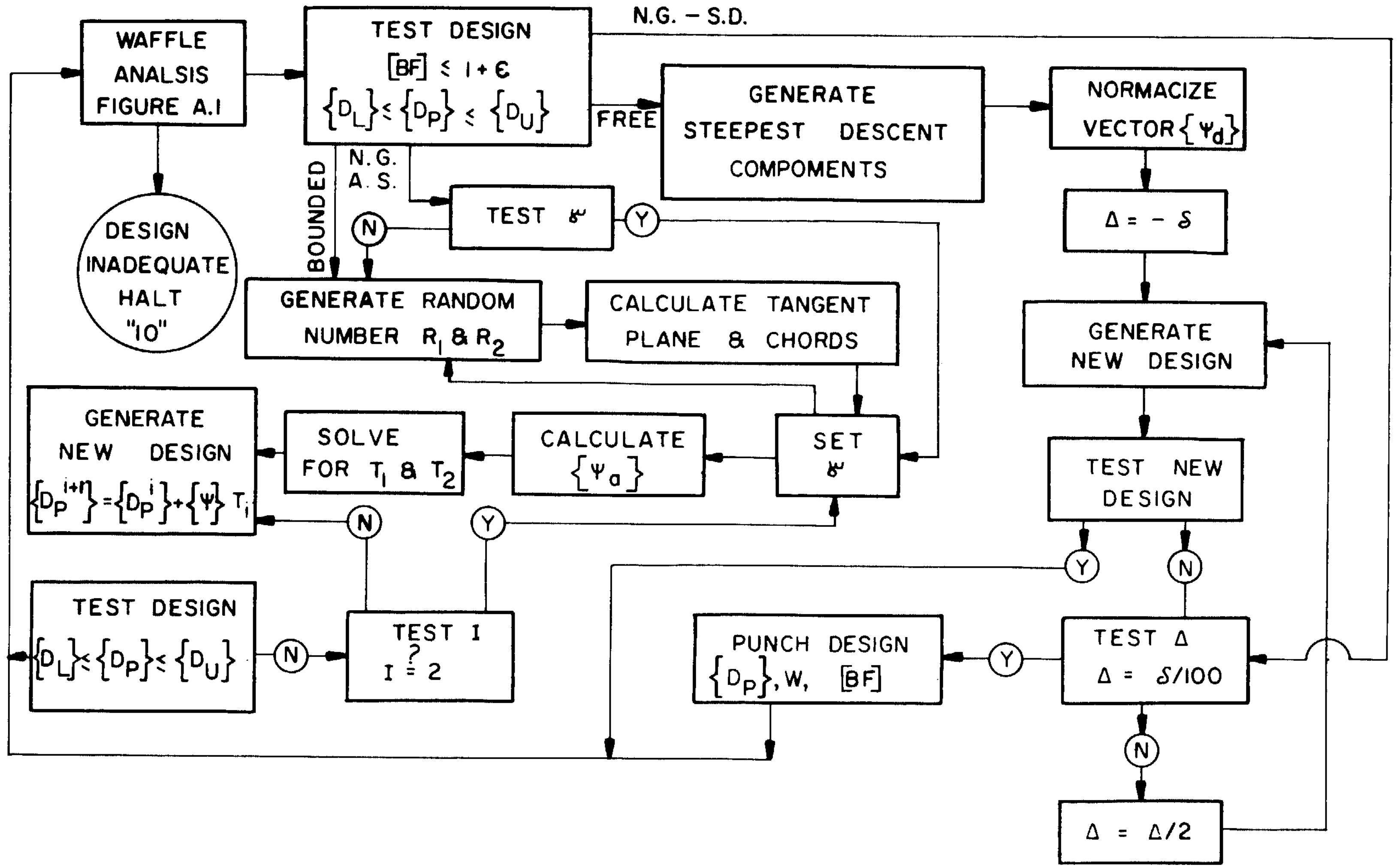


FIGURE II COMPROMISE II FLOW DIAGRAM